We first recall the definition of the Riemann integral (on the interval \([0, 1]\)). A partition, \(\mathcal{P}\), of the interval \([0, 1]\) is a finite increasing sequence \(x_0 < x_1 \cdots < x_n\) with \(x_0 = 0\) and \(x_1 = 1\). The partition \(\mathcal{P}\) divides the interval \([0, 1]\) into \(n\) subintervals \([x_{i-1}, x_i]\) of width \(\Delta_i = x_i - x_{i-1}\). Given a function 

\[ f : [0, 1] \rightarrow \mathbb{R} \]

for each subinterval define

\[ m_i = \inf_{x \in [x_{i-1}, x_i]} f(x) \]

and

\[ M_i = \sup_{x \in [x_{i-1}, x_i]} f(x). \]

We then define the lower and upper Riemann sums by

\[ L(f, \mathcal{P}) = \sum m_i \Delta_i \]

and

\[ U(f, \mathcal{P}) = \sum M_i \Delta_i. \]

The lower and upper Riemann integrals are then

\[ \int f = \sup L(f, \mathcal{P}) \]

and

\[ \int f = \inf U(f, \mathcal{P}). \]

We say that \(f\) is Riemann integrable if

\[ \int f = \int f. \]

For a Riemann integrable function we write

\[ \int f = \int f. \]

1. Show that a continuous function on \([0, 1]\) is Riemann integrable. You can do this however you like but below is an outline of a proof for which you can fill in the details.

   (a) A partition \(\mathcal{P}'\) is a refinement of \(\mathcal{P}\) if \(\mathcal{P}\) is contained in \(\mathcal{P}'\) as a set. Show that

   \[ L(f, \mathcal{P}) \leq L(\mathcal{P}') \]

   and

   \[ U(f, \mathcal{P}) \geq U(\mathcal{P}'). \]
(b) Show that for any two arbitrary partitions \( P \) and \( P' \) we have
\[
L(f, P) \leq U(f, P').
\]
(Hint: Look at the common partition \( P \cup P' \) of \( P \) and \( P' \).)

c) Use the fact that a continuous function on a compact interval is uniformly continuous
to show that for any \( \epsilon > 0 \) there exists a partition \( P \) with
\[
U(f, P) - L(f, P) \leq \epsilon.
\]
(d) Finish the proof!

2. Let \( \chi_{[a,b]} \) be the characteristic function of the interval \([a,b] \subseteq [0,1]\). Show that \( \chi_{[a,b]} \) is
Riemann integrable and that
\[
\int_{[0,1]} \chi_{[a,b]} = b - a.
\]

3. Let \( f \) be the function that is 1 on the rationals and 0 on the irrationals. Show that \( f \) is not
Riemann integrable.

4. For any \( \epsilon > 0 \) show that there exists a closed subset \( A \) of the interval \([0,1]\) whose interior is
empty but the Lesbesgue measure, \( m(A) \), of \( A \) is \( \geq 1 - \epsilon \). (Bonus: Show that there exists
such an \( A \) with the \( m(A) = 1 - \epsilon \).)