FOURIER–MUKAI TRANSFORMS COMMUTING WITH FROBENIUS

DANIEL BRAGG

ABSTRACT. We show that a Fourier–Mukai equivalence between smooth projective varieties of characteristic $p$ which commutes with either pushforward or pullback along Frobenius is a composition of shifts, isomorphisms, and tensor product with invertible sheaves of order $p - 1$.

1. Introduction

If $X$ is a smooth projective variety over a field $k$, we write $D^b(X)$ for the bounded derived category of coherent sheaves on $X$, viewed as a $k$–linear triangulated category. Let $p$ be a prime number. The absolute Frobenius morphism of a scheme $X$ over $F_p$ is the map $F_X : X \to X$ of schemes which is induced by the $p$th power map $\mathcal{O}_X \to \mathcal{O}_X$ given on local sections by $f \mapsto f^p$.

Let $k$ be an algebraically closed field of characteristic $p$ and let $X$ be a smooth projective variety over $k$. The Frobenius morphism $F_X$ is finite and flat, so both $F_X^*$ and $F_X^*$ are exact. We therefore obtain endofunctors

$$F_X^*, F_X^* : D^b(X) \to D^b(X)$$

We remark that these functors need not be $k$–linear; instead, they are $k$–semilinear with respect to the Frobenius morphism of $k$. Let $Y$ be another smooth projective variety over $k$, and let

$$\Phi : D^b(X) \to D^b(Y)$$

be a $k$–linear Fourier–Mukai equivalence. We say that $\Phi$ commutes with $F_*$ (resp. $\Phi$ commutes with $F^*$) if the diagram

$$
\begin{array}{ccc}
D^b(X) & \xrightarrow{\Phi} & D^b(Y) \\
F_X^* \downarrow & & \downarrow F_Y^* \\
D^b(X) & \xrightarrow{\Phi} & D^b(Y)
\end{array}
$$

commutes up to a natural isomorphism. Here are some examples of equivalences $\Phi$ which commute with both $F_*$ and $F^*$:

(1) the shift functor $[n] : D^b(X) \to D^b(X)$ for any $n \in \mathbb{Z}$,

(2) $f_* : D^b(X) \to D^b(Y)$, where $f : X \to Y$ is an isomorphism over $k$, and

(3) $\cdot \otimes L : D^b(X) \to D^b(X)$, where $L$ is an invertible sheaf on $X$ such that $L^\otimes (p-1) \cong \mathcal{O}_X$. 

1
Indeed, the shift functor commutes with any self-map of triangulated categories (by definition). The absolute Frobenius has the property that $F_Y \circ f = f \circ F_X$ for any map of schemes $f : X \to Y$. Finally, if $L$ is an invertible sheaf on $X$, then $F_X^* L \cong L^{\otimes p}$. Therefore if $L^{\otimes p^{-1}} \cong \mathcal{O}_X$, or equivalently $F_X^* L \cong L$, then for any $E \in D^b(X)$ we have isomorphisms

$$F_X^*(E \otimes L) \cong F_X^*E \otimes F_X^*L \cong F_X^*E \otimes L$$

and

$$F_{X*}(E \otimes L) \cong F_{X*}(E \otimes F_X^*L) \cong F_{X*}E \otimes L$$

which are functorial in $E$.

In this note we will show that these are in fact the only examples.

**Theorem 1.1.** If $\Phi : D^b(X) \to D^b(Y)$ is a Fourier–Mukai equivalence which commutes with $F_*$ or with $F^*$, then $\Phi$ is a composition of functors of the above form.

The key step is Proposition 3.3, which shows that we can characterize the shifts of structure sheaves of closed points of $X$ among all objects of $D^b(X)$ in terms of the $k$–linear triangulated category structure on $D^b(X)$ together with the Frobenius endofunctor $F_{X*} : D^b(X) \to D^b(X)$.

1.1. **Acknowledgements.** The question of which Fourier–Mukai equivalences commute with Frobenius was asked of the author by Karl Schwede.

2. **Equivalences preserving supports**

Let $X$ be a smooth variety over an algebraically closed field $k$ (of arbitrary characteristic). The *support* of a coherent sheaf $E$ on $X$ is the closed subscheme of $X$ cut out by the ideal sheaf $I_Z \subset \mathcal{O}_X$ defined as the kernel of the action map

$$\mathcal{O}_X \to \mathcal{E}nd_{\mathcal{O}_X}(E).$$

Equivalently, the support of $E$ is the minimal closed subscheme $Z \subset X$ such that $E$ is the pushforward of a coherent sheaf on $Z$. The *support* of a complex $E \in D^b(X)$ is the minimal closed subscheme of $X$ which contains the supports of the cohomology sheaves of $E$. If $E$ is a coherent sheaf or an object of $D^b(X)$, we define the *reduced support* of $E$ to be the reduced subvariety of $X$ underlying the support.

We make the following definition.

**Definition 2.1.** An object $E \in D^b(X)$ is point–like if

1. $\text{Hom}_{D^b(X)}(E, E[i]) = 0$ for $i < 0$ and
2. $\text{Hom}_{D^b(X)}(E, E) \cong k$.

If $x \in X$ is a closed point, then any shift $k(x)[n]$ is a point–like object of $D^b(X)$. In general, there may be point–like objects with positive dimensional support. The following result shows however that every point–like object with 0–dimensional support is of this form.
Lemma 2.2. If $E \in \mathcal{D}^b(X)$ is a point–like object with 0–dimensional support, then $E \cong k(x)[n]$ for some closed point $x \in X$ and integer $n$.

Proof. See [1, Lemma 4.5].

Suppose now that $X$ is smooth and projective. Let $Y$ be another smooth projective variety over $k$ and let $\Phi : \mathcal{D}^b(X) \to \mathcal{D}^b(Y)$ be a Fourier–Mukai equivalence with kernel $P \in \mathcal{D}^b(X \times Y)$.

Proposition 2.3. Suppose that, for every closed point $x \in X$, the support of the complex $\Phi(k(x))$ has dimension 0. Then the support of $P$ is the graph of an isomorphism $f : X \to Y$, and $\Phi$ is a composition of shifts, $f_*$, and tensoring with line bundles.

Proof. See [1, Corollary 5.23] (This has slightly stronger assumptions, but the proof is essentially the same).

3. Characterizing points using Frobenius

Let $X$ be a smooth variety over a field $k$. Let $Z \subset X$ be a reduced and irreducible closed subscheme. We say that a coherent sheaf $E$ on $X$ is properly supported on $Z$ if there is an irreducible component of the support of $E$ which contains $Z$ and has the same dimension as $Z$. Equivalently, $E$ is properly supported on $Z$ if $Z$ is an irreducible component of the reduced support of $E$. Suppose that $E$ is properly supported on $Z$. Let $Z'$ be the unique irreducible component of the support of $E$ which contains $Z$, let $E'$ be the restriction of $E$ to $Z'$, let $\eta$ be the generic point of $Z$, and let $\mathcal{O}_{X,\eta}$ be the local ring of $X$ at $\eta$. The support of $E'_\eta$ is then a closed subscheme of $\text{Spec} \mathcal{O}_{X,\eta}$ of dimension 0. It follows that the length of $E'_\eta$ as a module over $\mathcal{O}_{X,\eta}$ is finite. We define

$$\text{rk}_Z(E) := \text{len}_{\mathcal{O}_{X,\eta}}(E'_\eta).$$

The key property of this quantity that we will use is that it is additive in short exact sequences of coherent sheaves properly supported on $Z$.

Now let $X$ be a smooth variety over an algebraically closed field $k$ of characteristic $p > 0$. Let $Z \subset X$ be a reduced and irreducible closed subvariety of dimension $d$. Let $E$ be a coherent sheaf on $X$ which is properly supported on $Z$.

Lemma 3.1. We have

$$\text{rk}_Z(F_{X,\ast}E) = p^d \text{rk}_Z(E).$$

Proof. Let $i : Z \hookrightarrow X$ be the inclusion. It suffices to prove the result under the additional assumptions that $Z$ is smooth and irreducible and the support of $E$ is irreducible. We make these assumptions in the following. Let $Z'$ be the support of $E$.

We first consider the special case when $Z = Z' = X$. By shrinking $X$, we may even assume that $E$ is free. We may further reduce to the case when
\( E = \mathcal{O}_X \). The result then follows from the fact that the absolute Frobenius of a smooth \( k \)-variety of dimension \( n \) has degree \( p^n \).

Next we consider the special case when \( Z = Z' \). We have a commutative diagram

\[
\begin{array}{ccc}
Z & \xrightarrow{F_Z} & Z \\
\downarrow{i} & & \downarrow{i} \\
X & \xrightarrow{F_X} & X.
\end{array}
\]

This implies that

\[
F_X^* E = F_X^* i^* i^* E = i_*(F_{Z'}^*(i^* E))
\]

and hence

\[
i^*(F_X^* E) = i^* i_*(F_{Z'}^*(i^* E)) = F_{Z'}(i^* E).
\]

We have

\[
\text{rk}_Z(F_X^* E) = \text{rk}_Z(i^* F_X^* E) = \text{rk}_Z(F_{Z'}^*(i^* E)) = p^d \text{rk}_Z(i^* E) = p^d \text{rk}_Z(E)
\]

where the second to last equality is from the previous case.

Finally, we consider the general case. Let \( I \) be the ideal sheaf of \( Z \). We induct on the claimed statement for those coherent sheaves \( E \) which have irreducible support and are properly supported on \( Z \), and satisfy \( I^m E = 0 \). If \( IE = 0 \), then the support of \( E \) is contained in \( Z \), and the result follows from the previous case. This gives the base case. For the induction step, suppose that \( I^{m+1} E = 0 \), and consider the short exact sequence

\[
0 \to \text{gr}^m E \to E \to E/I^m E \to 0
\]

where \( \text{gr}^m(E) := I^m E/I^{m+1} E \). Applying \( F_X^* \) and using the additivity of the rank function, we get

\[
\text{rk}_Z(F_X^* E) = \text{rk}_Z(F_X^* \text{gr}^m E) + \text{rk}_Z(F_X^* (E/I^m E)) = p^d \text{rk}_Z(\text{gr}^m E) + p^d \text{rk}_Z(E/I^m E) = p^d \text{rk}_Z(E).
\]

\[ \square \]

Lemma 3.2. Let \( E \in \text{D}^b(X) \) be a nonzero bounded complex. If \( E \cong F_X^* E \), then \( E \) has 0-dimensional support.

Proof. If \( E \cong F_X^* E \), then also \( \mathcal{H}^n(E) \cong \mathcal{H}^n(F_X^* E) = F_X^* \mathcal{H}^n(E) \) for each integer \( n \). We therefore reduce to the case when \( E \) is a nonzero coherent sheaf. Let \( Z \) be the reduction of an irreducible component of the support of \( E \). We have

\[
\text{rk}_Z(E) = \text{rk}_Z(F_X^* E) = p^d \text{rk}_Z(E)
\]

where \( d \) is the dimension of \( Z \). It follows that \( d = 0 \). The result follows from Lemma 3.1. \( \square \)

Combining the above results, we obtain the following characterization of the shifts of structure sheaves of points in \( \text{D}^b(X) \).
Proposition 3.3. Let $E \in \text{D}^b(X)$ be an object. The following are equivalent.

1. $E \cong k(x)[n]$ for some closed point $x \in X$ and integer $n$.
2. $E$ is point-like and $E \cong F_{X*}E$.

Proof. (1) implies (2) is immediate. We prove (2) implies (1). By Lemma 3.2, if $E \cong F_{X*}E$ then $E$ has 0-dimensional support. By Lemma 2.2, we conclude that $E \cong k(x)[n]$, as claimed. □

4. Proof of Theorem 1.1

We recall the notation: $X$ and $Y$ are smooth projective varieties over an algebraically closed field $k$ of characteristic $p > 0$, and $\Phi : \text{D}^b(X) \to \text{D}^b(Y)$ is a Fourier–Mukai equivalence.

Lemma 4.1. The equivalence $\Phi$ commutes with $F^*$ if and only if it commutes with $F_*^*$.

Proof. We have the adjunction

$$\text{Hom}_{\text{D}^b(X)}(F_X^*E, G) = \text{Hom}_{\text{D}^b(X)}(E, F_{X*}G)$$

for objects $E, G \in \text{D}^b(X)$. Because $\Phi$ is an equivalence, this gives rise to isomorphisms

$$\text{Hom}_{\text{D}^b(Y)}(\Phi(F_X^*E), \Phi(G)) = \text{Hom}_{\text{D}^b(Y)}(\Phi(E), \Phi(F_{X*}G))$$

which are functorial in $E$ and $G$. Suppose that $F_Y^* \circ \Phi \cong \Phi \circ F_X^*$. Then we obtain functorial isomorphisms

$$\text{Hom}_{\text{D}^b(Y)}(\Phi(E), F_{Y*}\Phi(G)) = \text{Hom}_{\text{D}^b(Y)}(F_Y^*\Phi(E), \Phi(G))$$

$$= \text{Hom}_{\text{D}^b(Y)}(\Phi(F_X^*E), \Phi(G))$$

$$= \text{Hom}_{\text{D}^b(Y)}(\Phi(E), \Phi(F_{X*}G)).$$

As $\Phi$ is an equivalence, we conclude that $F_{Y*} \circ \Phi \cong \Phi \circ F_X^*$. The reverse implication is similar. □

Proof of Theorem 1.1. By Lemma 4.1, it suffices to consider the case when $\Phi$ commutes with $F_*^*$. For a closed point $x \in X$, we have that $\Phi(k(x))$ is point-like, and furthermore

$$\Phi(k(x)) = F_{X*}\Phi(k(x)) = F_{X*}F_X^*k(x) = \Phi(F_X^*k(x))$$

Proposition 3.3 implies that $\Phi(k(x)) \cong k(y)[n]$ for some closed point $y \in Y$ and some integer $n$. By Proposition 2.3, there exists an isomorphism $f : X \to Y$, an integer $n$, and a line bundle $L$ on $X$ such that $\Phi$ is given by

$$\Phi(E) = f_* (E \otimes L)[n]$$

It remains to show that $L^\otimes p^{-1} \cong \mathcal{O}_X$. To see this, we note that shifts and pushforwards along isomorphisms always commute with both $F^*$ and $F_*$. This therefore implies that tensoring with $L$ commutes with $F_*^*$, and hence also with $F^*$. We have

$$L \otimes F^*E \cong F^*(L \otimes E)$$
for every $E \in D^b(X)$. In particular, taking $E = \mathcal{O}_X$ we conclude that $L^\otimes p^{-1} \cong \mathcal{O}_X$. \hfill \Box

References


Department of Mathematics, University of Utah, Salt Lake City, UT 84112
E-mail address: bragg@math.utah.edu