1. Introduction

The goal of this note is to prove that continuous functions are Riemann integrable. We recall what this means:

Definition 1.1. Let f be a real–valued function defined on a domain $D \subset \mathbb{R}$. Let $[a, b]$ be a closed bounded interval contained in D. We say that f is *Riemann integrable* on $[a, b]$ if f is bounded on $[a, b]$ and

$$\underline{\int_a^b} f(x)\,\mathrm{d}x = \overline{\int_a^b} f(x)\,\mathrm{d}x.$$

2. A differential analog of induction

I like to think of the derivative of a function as a continuous analog of the *discrete difference operator*, defined as follows. Say $g : \mathbb{N} \to \mathbb{R}$ is a sequence of real numbers. We set

$$Dg(x) = f(x + 1) - f(x).$$

If you plot the points $(x, g(x))$ for $x \in \mathbb{N}$, then $Dg(x)$ is the slope of the line segment between $(x, g(x))$ and $(x + 1, g(x + 1))$. The following is a rephrasing of induction.

Lemma 2.1. *Suppose that g and h are two functions from $\mathbb{N}$ to $\mathbb{R}$. Then $g(x) = h(x)$ for all $x \in \mathbb{N}$ if and only if the following two conditions hold:*
(1) $g(1) = h(1)$.
(2) $Dg(x) = Dh(x)$ for all $x \in \mathbb{N}$.

Proof. Suppose that the two conditions hold. We will prove that $g(x) = h(x)$ for all $x \in \mathbb{N}$ by induction on x. The base case of $x = 1$ is true by assumption. Suppose that we already know $g(x) = h(x)$ for some x. We assume $Dg(x) = Dh(x)$, or in other words $g(x + 1) - g(x) = h(x + 1) - h(x)$. Adding $g(x) = h(x)$, we get $g(x + 1) = h(x + 1)$. The result follows by induction. □

Here is the differential analog of the above.

Lemma 2.2. *Let $[a, b]$ be a closed bounded interval, and let g, h be functions defined on $[a, b]$. Then $g(x) = h(x)$ for all $x \in [a, b]$ if the following two conditions hold:*
(1) $g(a) = h(a)$.
(2) g and h are differentiable on (a, b) and $g'(x) = h'(x)$ for all $x \in (a, b)$.

3. Integrability of continuous functions

Theorem 3.1. *If f is a continuous function defined on a closed bounded interval $[a, b]$, then f is Riemann integrable on $[a, b]$.*

Proof. As f is continuous on $[a, b]$, it is also bounded on $[a, b]$. Thus, to show that f is Riemann integrable on $[a, b]$ we need to show that

$$\underline{\int_a^b} f(x)\,\mathrm{d}x = \overline{\int_a^b} f(x)\,\mathrm{d}x.$$

For $x \in [a, b]$, we define

$$I(x) = \overline{\int_a^x} f(t)\,\mathrm{d}t \text{ and}$$
$$i(x) = \underline{\int_a^x} f(t)\,\mathrm{d}t.$$

We will show that $I(x) = i(x)$ for all $x \in [a, b]$. In particular, this will imply that $I(b) = i(b)$, so this will give what we want. We will check the conditions of Lemma 2.2. We have $I(a) = i(a) = 0$, so the first condition holds. We check the second condition. Fix a point $x \in (a,b)$ and an $\varepsilon > 0$. As f is continuous, there exists a $\delta > 0$ such that if $|x-y| < \delta$ then $|f(x) - f(y)| < \varepsilon$. We may assume that δ is small enough so that we have $a < x - \delta < x < x + \delta < b$. Let h be a positive real number such that $h < \delta$. Consider the partition $P_h = \{x, x+h\}$ of the interval $[x, x+h]$. We have the inequalities

$$L(f, P_h) \le \underline{\int_x^{x+h}} f(t)\,\mathrm{d}t \le \overline{\int_x^{x+h}} f(t)\,\mathrm{d}t \le U(f, P_h).$$

Set

$$M_h = \sup_{[x,x+h]} f \qquad \text{and} \qquad m_h = \inf_{[x,x+h]} f.$$

We know that

$$f(x) - \varepsilon \le m_h \le M_h \le f(x) + \varepsilon.$$

The upper and lower Riemann sums for f on P_h are given by

$$U(f, P_h) = M_h \cdot h \qquad \text{and} \qquad L(f, P_h) = m_h \cdot h.$$

We also know that the upper and lower integrals are additive, so we have

$$I(x+h) - I(x) = \overline{\int_x^{x+h}} f(t)\,\mathrm{d}t \qquad \text{and} \qquad i(x+h) - i(x) = \underline{\int_x^{x+h}} f(t)\,\mathrm{d}t.$$

Putting these together, we get the inequalities

$$(f(x) - \varepsilon) \cdot h \le m_h \cdot h \le i(x+h) - i(x) \le I(x+h) - I(x) \le M_h \cdot h \le (f(x) - \varepsilon) \cdot h.$$

Dividing by h, we get the inequalities

$$f(x) - \varepsilon \le m_h \le \frac{i(x+h) - i(x)}{h} \le \frac{I(x+h) - I(x)}{h} \le M_h \le f(x) + \varepsilon.$$

To recap, we have show that that for any $\varepsilon > 0$, there exists a $\delta > 0$ such that $a < x - \delta < x < x + \delta < b$ and for any h such that $0 < h < \delta$, we have the above chain of inequalities. It follows that

$$\lim_{h \to 0^+} \frac{i(x+h) - i(x)}{h} = \lim_{h \to 0^+} \frac{I(x+h) - I(x)}{h} = f(x).$$

A similar argument involving a partition of the interval $[x-h, x]$ shows that

$$\lim_{h \to 0^-} \frac{i(x+h) - i(x)}{h} = \lim_{h \to 0^-} \frac{I(x+h) - I(x)}{h} = f(x).$$

We conclude that

$$\lim_{h \to 0} \frac{i(x+h) - i(x)}{h} = \lim_{h \to 0} \frac{I(x+h) - I(x)}{h} = f(x).$$

Thus, I and i are both differentiable at x, and we have

$$i'(x) = I'(x) = f(x).$$

□

4. Some stronger results

For the record, I'm recording here some stronger results on integrability. We proved the following in lecture. I'm not including the proof here.

Theorem 4.1. *Let f be a bounded function defined on a closed bounded interval $[a,b]$. If f is continuous on $[a,b]$ away from finitely many points of $[a,b]$, then f is Riemann integrable on $[a,b]$.*

The following example shows that integrability doesn't need to hold if f has infinitely many discontinuities.

Example 4.2. Define a function f by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

Then f is discontinuous at *every* point in $\mathbb{R}$. We showed in class that f is *not* Riemann integral on any interval.

In fact, we can say something even when f has infinitely many discontinuities.

Definition 4.3. Let f be a function defined on an interval $[a, b]$. Let $c \in [a, b]$ be a point at which f is not continuous. We say that f has an *isolated discontinuity* at c if there exists an $\epsilon > 0$ such that f is continuous everywhere on the open interval $(c - \epsilon, c + \epsilon)$ except at c itself.

Theorem 4.4. *Let f be a bounded function defined on a closed bounded interval $[a, b]$. If all but finitely many of the discontinuities of f in the interval $[a, b]$ are isolated, then f is Riemann integrable on $[a, b]$.*

Here is an example of a function which has infinitely many isolated discontinuities.

Example 4.5. Consider the "square wave" function

$$f(x) = \operatorname{sgn}(\sin \pi x).$$

Here, $\operatorname{sgn}(y)$ is the "sign" function, defined by

$$\operatorname{sgn}(y) = \begin{cases} 1 & \text{if } y > 0 \\ 0 & \text{if } y = 0 \\ -1 & \text{if } y < 0. \end{cases}$$

The square wave $f(x)$ jumps between the values 1 and -1. It is discontinuous at every point where $\sin \pi x = 0$, or in other words at every integer $x \in \mathbb{Z}$. Now consider the modified square wave

$$g(x) = \begin{cases} f\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Then g is discontinuous at the points $x = 0$ and $x = \frac{1}{n}$ for n a nonzero integer. So, for instance, g is discontinuous at infinitely many points in any interval $[0, a]$. We note that the point $x = 0$ is a non–isolated discontinuity, and each of the points $x = \frac{1}{n}$ is an isolated discontinuity. Thus, Theorem 4.4 implies that g is nevertheless Riemann integrable on $[0, a]$ (in fact, on any closed bounded interval).