Review for final exam (Math 3210, Fall 2023)

1 Solutions to sample problems

1. Consider a partition $P = \{x_0, x_1, \ldots, x_n\}$ of the interval $[a, b]$, and as usual set $I_k = [x_{k-1}, x_k]$. Because $f(x) = C$ is constant, we have

$$\sup_{I_k} f = C \quad \text{and} \quad \inf_{I_k} f = C.$$

Thus, the upper and lower sums of f with respect to P are given by

$$U(f, P) = \sum_{k=1}^{n} C\,(x_k - x_{k-1}) = C(b - a) \quad \text{and}$$
$$L(f, P) = \sum_{k=1}^{n} C\,(x_k - x_{k-1}) = C(b - a).$$

Therefore the upper and lower integrals are given by

$$\overline{\int_a^b} f(x)\,\mathrm{d}x \overset{\text{def}}{=} \inf_P U(f, P) = C(b - a) \quad \text{and}$$
$$\underline{\int_a^b} f(x)\,\mathrm{d}x \overset{\text{def}}{=} \sup_P L(f, P) = C(b - a).$$

Thus, we have

$$\overline{\int_a^b} f(x)\,\mathrm{d}x = C(b - a) = \underline{\int_a^b} f(x)\,\mathrm{d}x.$$

We have shown that f is Riemann integrable on $[a, b]$, and that its integral is

$$\int_a^b f(x)\,\mathrm{d}x = C(b - a).$$

2. The upper sum is

$$U(f, P) = \sum_{k=1}^{4} M_k(x_k - x_{k-1}) = \frac{1}{1} \cdot 1 + \frac{1}{2} \cdot 1 + \frac{1}{3} \cdot 1 = \frac{11}{6} \sim 1.833.$$

The lower sum is

$$L(f, P) = \sum_{k=1}^{4} m_k(x_k - x_{k-1}) = \frac{1}{2} \cdot 1 + \frac{1}{3} \cdot 1 + \frac{1}{4} \cdot 1 = \frac{13}{12} \sim 1.0833.$$

3. We always have

$$L(f, P) \leq \int_a^b f(x)\,\mathrm{d}x \leq U(f, P)$$

for any partition P. Thus, our answer from the previous problem tells us that

$$\frac{13}{12} \leq \int_1^4 \frac{1}{x}\,\mathrm{d}x \leq \frac{11}{6}.$$

As a consistency check, let's note that we actually know how to compute the integral exactly: the true value of the integral is given by

$$\int_1^4 \frac{1}{x}\,\mathrm{d}x = \ln 4 - \ln 1 \sim 1.386.$$

We have

$$1.0833 < 1.386 < 1.833$$

so our answer makes sense.

4. We know that f is bounded on $[a, b]$, so we can find $m, M \in \mathbb{R}$ such that $m \leq f(x) \leq M$ for all $x \in [a, b]$. Fix a real number r such that $a \leq r < b$. By Exercise 5.1.8, we have

$$m(b-r) \leq \underline{\int_r^b} f(x)\,\mathrm{d}x \leq \overline{\int_r^b} f(x)\,\mathrm{d}x \leq M(b-r).$$

It follows that

$$\overline{\int_r^b} f(x)\,\mathrm{d}x - \underline{\int_r^b} f(x)\,\mathrm{d}x \leq (M-m)(b-r) \tag{1.1}$$

By Theorem 5.2.8, we have

$$\overline{\int_a^b} f(x)\,\mathrm{d}x = \overline{\int_a^r} f(x)\,\mathrm{d}x + \overline{\int_r^b} f(x)\,\mathrm{d}x$$

and

$$\underline{\int_a^b} f(x)\,\mathrm{d}x = \underline{\int_a^r} f(x)\,\mathrm{d}x + \underline{\int_r^b} f(x)\,\mathrm{d}x.$$

We know that $f(x)$ is integrable on $[a, r]$, so we have

$$\overline{\int_a^r} f(x)\,\mathrm{d}x = \underline{\int_a^r} f(x)\,\mathrm{d}x.$$

Thus, subtracting the above two equations and using (1.1), we get that

$$\overline{\int_a^b} f(x)\,\mathrm{d}x - \underline{\int_a^b} f(x)\,\mathrm{d}x = \overline{\int_b^r} f(x)\,\mathrm{d}x - \underline{\int_b^r} f(x)\,\mathrm{d}x \leq (M-m)(b-r)$$

for every $r \in [a, b)$. This shows that the left hand difference is smaller than every positive real number, and hence

$$\overline{\int_a^b} f(x)\,\mathrm{d}x - \underline{\int_a^b} f(x)\,\mathrm{d}x = 0.$$

Thus, the upper and lower integrals of f on $[a,b]$ are equal, so f is integrable on $[a,b]$. Furthermore, we have that

$$\int_a^b f(x)\,\mathrm{d}x - \int_a^r f(x)\,\mathrm{d}x = \int_r^b f(x)\,\mathrm{d}x \le M(b-r)$$

for any $r \in [a,b)$. This shows that

$$\lim_{r\to b}\left(\int_a^b f(x)\,\mathrm{d}x - \int_a^r f(x)\,\mathrm{d}x\right) \le \lim_{r\to b} M(b-r) = 0$$

and hence

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{r\to b}\int_a^r f(x)\,\mathrm{d}x.$$

5. If $-1 \le x \le 1$, then $0 \le x^{2n} \le 1$. Thus, $1 \le 1 + x^{2n} \le 2$, and so we have

$$\frac{1}{2} \le \frac{1}{1+x^{2n}} \le 1.$$

By Corollary 5.2.5, we get

$$1 = \frac{1}{2}\cdot 2 \le \int_{-1}^1 \frac{1}{1+x^{2n}}\,\mathrm{d}x \le 1\cdot 2 = 2.$$

6. We will show that the right hand side is the least upper bound for the set

$$S \overset{\text{def}}{=} \{|f(x)-f(y)| | x,y \in I\}\,.$$

Let's first show that it is an upper bound. If $x,y \in I$, then $f(x) \le \sup_I f$ and $f(y) \ge \inf_I f$, hence $-f(y) \le -\inf_I f$. Thus, we get

$$f(x) - f(y) \le \sup_I f - \inf_I f.$$

By the same reasoning, we get

$$-(f(x)-f(y)) = f(y) - f(x) \le \sup_I f - \inf_I f,$$

and therefore

$$|f(x) - f(y)| \le \sup_I f - \inf_I f.$$

Thus, the right hands side is an upper bound, so we get

$$\sup_{x,y\in I} |f(x)-f(y)| \le \sup_{x\in I} f(x) - \inf_{x\in I} f(x).$$

Next we will show that the right hand side is in fact the *least* upper bound for S, or in other words that this inequality is in fact an equality. Suppose that $a \in \mathbb{R}$ is a real number such that

$$a < \sup_{x\in I} f(x) - \inf_{x\in I} f(x).$$

We will show that a is not an upper bound for S, or in other words that there exist $x,y \in I$ such that

$$a < |f(x) - f(y)|.$$

To prove this, we rearrange the above to get

$$a + \inf_I f < \sup_I f.$$

As $\sup_I f$ is the *least* upper bound for the set $\{f(x)|x \in I\}$, this means that there exists an $x \in I$ such that

$$a + \inf_I f < f(x).$$

Now rearrange this to get

$$f(x) - a > \inf_I f.$$

As $\inf_I f$ is the *greatest* lower bound for the set $\{f(x)|x \in I\}$, this means that there exists a $y \in I$ such that

$$f(x) - a > f(y).$$

It follows that

$$a < f(x) - f(y) \leq |f(x) - f(y)|,$$

as claimed. We have shown that $\sup_{x\in I} f(x) - \inf_{x\in I} f(x)$ is the least upper bound for S, and so conclude that

$$\sup_{x,y\in I} |f(x) - f(y)| = \sup_{x\in I} f(x) - \inf_{x\in I} f(x).$$

7. By the second fundamental theorem of calculus (Theorem 5.3.3), we have

$$\frac{\mathrm{d}}{\mathrm{d}x} \int_1^x \cos 1/t \; \mathrm{d}t = \cos 1/x.$$

8. Say f and g are two functions on $[a, b]$, that f is integrable on $[a, b]$, and that $c \in [a, b]$ is a point such that $f(x) = g(x)$ for all $x \in [a, b]$ except possibly $x = c$. For any $r \in [a, c)$, we have that $f = g$ everywhere on the interval $[a, r]$, so g is integrable on $[a, r]$ and

$$\int_a^r g(x)\,\mathrm{d}x = \int_a^r f(x)\,\mathrm{d}x.$$

By problem (**4**), we get that g is integrable on $[a, c]$, and moreover

$$\int_a^c g(x)\,\mathrm{d}x = \lim_{r\to c} \int_a^r g(x)\,\mathrm{d}x = \lim_{r\to c} \int_a^r f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x.$$

By the same reasoning, we get that g is integrable on $[c, b]$, and moreover

$$\int_c^b g(x)\,\mathrm{d}x = \int_c^b f(x)\,\mathrm{d}x.$$

By Corollary 5.2.9, g is integrable on $[a, b]$, and the integral is given by

$$\int_a^b g(x)\,\mathrm{d}x = \int_a^c g(x)\,\mathrm{d}x + \int_c^b g(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x$$

9. The function f is given more explicitly by

$$f(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0. \end{cases}$$

By problem (**4**), we have that if $a < 0$ then

$$\int_a^0 f(x)\,\mathrm{d}x = \lim_{r \to 0^-} \int_a^r (-1)\,\mathrm{d}x = \lim_{r \to 0} (-r + a) = a.$$

Similarly, we get that if $b > 0$ then

$$\int_0^b f(x)\,\mathrm{d}x = b.$$

Putting these together, we can compute the integral when $a < 0 < b$: the answer is $b + a$. There are two remaining cases: when $0 < a < b$, the integral is $b - a$, and when $a < b < 0$, the integral is $-b + a$. A nice way to put all these cases together into one formula is $|b| - |a|$ (check that this does indeed agree with our answers in every case!). We conclude that

$$\int_a^b f(x)\,\mathrm{d}x = |b| - |a|$$

for any any real numbers $a < b$.

10. If $x > 0$, then $|x| = x$, so the function is given by

$$F(x) = 1 + x.$$

Thus $F'(x) = 1 = f(x)$. If $x < 0$, then $|x| = -x$, so the function is given by

$$F(x) = -1 - x.$$

Thus $F'(x) = -1 = f(x)$. However, we have that

$$\int_{-1}^1 f(x)\,\mathrm{d}x = |1| - |-1| = 0,$$

while

$$F(1) - F(-1) = 2 - (-1 + 1) = 2.$$

The reason this does not contradict Theorem 5.3.1 (the first fundamental theorem of calculus) is that F is *not* continuous on the interval $[-1, 1]$. Indeed, we have

$$\lim_{x \to 0^-} F(x) = -1$$

while

$$\lim_{x \to 0^+} F(x) = 1,$$

so $F(x)$ is not continuous at 0.