Review for Final Exam (Math 3210, Fall 2023)

1 Sample Problems

The final exam is cumulative, but will be weighted towards the content we've covered after midterm 3. These are sample problems only for the content we've covered *after* midterm 3.

1. Let $C \in \mathbb{R}$ be a number and let $a < b$ be two real numbers. Prove *directly from the definition of the Riemann integral* that the constant function $f(x) = C$ is integrable on $[a, b]$, and that

$$\int_a^b C \, \mathrm{d}x = C(b-a).$$

2. Consider the function $f(x) = 1/x$ on the interval $[1, 4]$. Compute the upper and lower sums $U(f, P)$ and $L(f, P)$, where P is the partition $P = \{1, 2, 3, 4\}$ of $[1, 4]$.

3. Using your answer from question (**2**), find upper and lower bounds for the integral $\int_1^4 \frac{1}{x} \, \mathrm{d}x$.

4. If f is a bounded function defined on a closed bounded interval $[a, b]$ and if f is integrable on each interval $[a, r]$ with $a < r < b$, then prove that f is integrable on $[a, b]$, and furthermore that

$$\int_a^b f(x) \, \mathrm{d}x = \lim_{r \to b} \int_a^r f(x) \, \mathrm{d}x.$$

   Hint: Use Theorem 5.2.8 and Exercise 5.1.8 from the book.

5. Prove that

$$1 \leq \int_{-1}^1 \frac{1}{1 + x^{2n}} \, \mathrm{d}x \leq 2$$

   for any $n \in \mathbb{N}$. **Hint**: use Corollary 5.2.5 from the book.

6. Let f be a bounded function on an interval I. Prove that

$$\sup_{x,y \in I} |f(x) - f(y)| = \sup_{x \in I} f(x) - \inf_{x \in I} f(x).$$

7. Fix a real number $x > 0$. Compute

$$\frac{\mathrm{d}}{\mathrm{d}x} \int_1^x \cos 1/t \, \mathrm{d}t.$$

8. Prove that if f is integrable on a closed bounded interval $[a, b]$ and $c \in [a, b]$, then changing the value of f at c does not change the fact that f is integrable or the value of its integral on $[a, b]$[1].

[1] In case this is confusing, here is an equivalent formulation of this question: suppose that f and g are two functions on $[a, b]$, and that $f(x) = g(x)$ for all $x \in [a, b]$ except possibly for $x = c$. Then show that f is integrable on $[a, b]$ if and only if g is integrable $[a, b]$, and that if this is the case then $\int_a^b f(x) \, \mathrm{d}x = \int_a^b g(x) \, \mathrm{d}x$.

9. Consider the function
$$f(x) = \begin{cases} \frac{x}{|x|} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$
Then f is bounded and is continuous away from a single point 0, so by a theorem we proved in lecture, f is integrable on any closed bounded interval $[a, b]$. Given real numbers a, b, compute the integral
$$\int_a^b f(x)\, \mathrm{d}x$$
in terms of a and b. **Hint**: try to guess what the answer should be by drawing a picture of the graph of f. Then prove your answer is correct by using problem (**4**).

10. Let f be the function from problem (**9**). Define a function $F(x)$ by
$$F(x) = \begin{cases} \frac{x}{|x|}(1 + x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$
Show that $F(x)$ is differentiable away from 0, and that $F'(x) = f(x)$ for $x \neq 0$. Using your answer from question (**9**), show however that
$$\int_{-1}^{1} f(x)\, \mathrm{d}x \neq F(1) - F(-1).$$
Explain why this does *not* contradict the First Fundamental Theorem of Calculus (Theorem 5.3.1 in the book).