Midterm 3 Math 3210 Fall 2023

Directions:

- You have 50 minutes to complete this exam.
- You may cite results proved during lecture or in the book without repeating the proof.
- If you wish to use a result from lecture or from the book, you must write out the complete *statement* of the result which you are citing.

Question	Points	Score
1	10	
2	10	
3	10	
Total:	30	

1. Consider the function f defined by

$$f(x) = \begin{cases} \sin x & \text{if } x \geq 0 \\ x + x^2 & \text{if } x < 0. \end{cases}$$

(a) (5 points) Prove that f is differentiable at 0 (you may use without proof that $\sin' x = \cos x$ and $\cos' x = -\sin x$).

> **Solution**: We need to prove that the limit $\lim_{x\to 0} f(x)/x$ exists (note that $f(0) = 0$). We will do this by computing both one–sided limits. We have
>
> $$\lim_{x\to 0^-} \frac{f(x)}{x} = \lim_{x\to 0^-} \frac{x + x^2}{x} = \lim_{x\to 0^-} (1 + x) = 1.$$
>
> We know that $\sin'(0) = 1$, so $\lim_{x\to 0} \sin x/x = 1$, and therefore
>
> $$\lim_{x\to 0^+} \frac{f(x)}{x} = \lim_{x\to 0^+} \frac{\sin x}{x} = 1.$$
>
> As both one–sided limits exist and are equal, the two–sided limit also exists. Therefore f is differentiable at 0. Note we have also shown that $f'(0) = 1$.

(b) (5 points) Prove that f is **not** twice differentiable at 0 (that is, that $f'(x)$ is not differentiable at 0).

> **Solution**: The derivative of f is given by the formula
>
> $$f'(x) = \begin{cases} \cos x & \text{if } x \geq 0 \\ 1 + 2x & \text{if } x < 0. \end{cases}$$
>
> We compute that
>
> $$\lim_{x\to 0^-} \frac{f'(x) - f'(0)}{x - 0} = \lim_{x\to 0^-} \frac{f'(x) - 1}{x} = \lim_{x\to 0^-} \frac{1 + 2x - 1}{x} = 2.$$
>
> On the other hand, we know that $\cos' 0 = 0$, so $\lim_{x\to 0} \frac{\cos x - 1}{x} = 0$, and therefore
>
> $$\lim_{x\to 0^+} \frac{f'(x) - f'(0)}{x - 0} = \lim_{x\to 0^+} \frac{f'(x) - 1}{x} = \lim_{x\to 0^+} \frac{\cos x - 1}{x} = 0.$$
>
> The two one–sided limits do not agree, so $\lim_{x\to 0} \frac{f'(x) - f'(0)}{x - 0}$ does not exist, and therefore f' is not differentiable at 0.

2. (10 points) Let f be a continuous real valued function defined on all of $\mathbb{R}$. Suppose that f is <u>bounded</u>, that is, that there exists a real number M such that $|f(x)| \leq M$ for all $x \in \mathbb{R}$. Prove that the function

$$g(x) = \begin{cases} xf(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is continuous at every point of $\mathbb{R}$.

Solution: We know that $1/x$ is continuous on $\mathbb{R} \setminus 0$. By a homework problem, the composition of continuous functions is continuous, so $f(1/x)$ is continuous on $\mathbb{R} \setminus 0$. By the main limit theorem, $xf(1/x)$ is continuous on $\mathbb{R} \setminus 0$. Thus, $g(x)$ is continuous at every point of $\mathbb{R} \setminus 0$. So, it remains only to prove that $g(x)$ is continuous at 0. By Theorem 1 in the theorem bank, $g(x)$ is continuous at 0 if and only if

$$\lim_{x \to 0} g(x) = g(0) = 0.$$

So, we need to prove that $\lim_{x \to 0} xf(1/x) = 0$. We will prove this directly from the definition of a limit. Let M be a number such that $|f(x)| \leq M$ for all $x \in \mathbb{R}$. We may assume that $M > 0$. Fix $\epsilon > 0$. If $x \neq 0$ and $|x| < \epsilon/M$, then

$$|g(x) - g(0)| = |xf(1/x)| \leq |x|M < \epsilon.$$

Therefore $\lim_{x \to 0} g(x) = g(0)$, so g is continuous at 0.

3. (10 points) Let $f(x)$ and $g(x)$ be real–valued functions which are defined and differentiable on all of $\mathbb{R}$. Suppose that $f(0) = g(0) = 0$. Show that if $f'(x) \leq g'(x)$ for all $x \geq 0$, then $f(x) \leq g(x)$ for all $x \geq 0$. **Hint**: use the Mean Value Theorem.

Solution: Consider the function $h(x) = f(x) - g(x)$. Because $f(0) = g(0)$, we have that $h(0) = 0$. Applying the Mean Value Theorem to the function $h(x) = f(x) - g(x)$ on the interval $[0, x]$, we get that there exists a $c > 0$ such that

$$h'(c) = \frac{h(x) - h(0)}{x - 0} = \frac{h(x)}{x}.$$

We know that $h'(c) = f'(c) - g'(c) \leq 0$ for all c, so $h'(c) \leq 0$. As $x > 0$, this implies that $h(x) \leq 0$. Therefore $f(x) - g(x) \leq 0$, so $f(x) \leq g(x)$. As x was arbitrary, this shows that $f(x) \leq g(x)$ for any $x > 0$. We also know by assumption that $f(0) = g(0)$, so $f(x) \leq g(x)$ for any $x \geq 0$.

Definitions and theorems:

Definition 1 *A function f defined on a domain $D \subset \mathbb{R}$ is continuous at a point $c \in D$ if, for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $x \in D$ and $|x - c| < \delta$, then $|f(x) - f(c)| < \epsilon$.*

Definition 2 *A function f defined on an open interval (a, b) is differentiable at a point $c \in (a, b)$ if the limit*

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

converges. In this case, we write $f'(c)$ for the above limit.

Theorem 1 *Let f be a function defined on a domain $D \subset \mathbb{R}$. Then f is continuous at a point $c \in D$ if and only if*

$$\lim_{x \to c} f(x) = f(c).$$

Theorem 2 (The Mean Value Theorem) *Suppose that f is a function defined and continuous on a closed interval $[a, b]$ which is differentiable on (a, b). There exists at least one point $c \in (a, b)$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$