Review for Midterm 3 (Math 3210, Fall 2023)

1 Solutions to sample problems

1. Consider a point $a \in \mathbb{R}$. We will show that f is continuous at a. Fix $\epsilon > 0$. Let $\delta = \epsilon$. If $|x - a| < \delta$, then we have

$$|f(x) - f(a)| = |(2|x| - 1) - (2|a| - 1)| = 2\big||x| - |a|\big| \leq |x - a| < \epsilon$$

where we have used the triangle inequality

$$\big||x| - |a|\big| \leq |x - a|.$$

Therefore f is continuous at a.

2. We know that 0 and x^2 are each continuous everywhere, so the function f is continuous at every point except possibly 0. To check that it is continuous at 0, we will use the result we proved in lecture that says that f is continuous at a point a if and only if

$$f(a) = \lim_{x \to a} f(x).$$

We compute $\lim_{x \to a} f(x)$ by first computing the two one–sided limits. We have

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x^2 = 0$$

and

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} 0 = 0.$$

As both one–sided limits exist and agree with each other, the two–sided limit also exists, and agrees with the one–sided limits. Thus, we have

$$\lim_{x \to 0} f(x) = 0$$

which is indeed equal to $f(0)$.

3. We note that

$$|\sin 1/x| \leq 1$$

for all x. Thus,

$$|f(x)| = |x \sin 1/x| \leq |x|$$

for all $x \in \mathbb{R} \setminus 0$. It follows that

$$\lim_{x \to 0} f(x) = 0.$$

4. We know that x, $1/x$, and $\sin x$ are continuous on their domains of definition, so $x \sin 1/x$ is continuous away from 0, and thus $f(x)$ is continuous at every point, except possibly at 0. We computed in the above problem that $\lim_{x\to 0} f(x) = f(0)$, so f is also continuous at 0.

5. We know that $\lim_{x\to\infty} \frac{1}{x} = 0$ and $\lim_{x\to\infty} \frac{1}{x^2} = 0$. Using the main limit theorem, we compute

$$\lim_{x\to\infty} \frac{x^2 - 3x - 1}{2x^2 + 5} = \lim_{x\to\infty} \frac{1 - 3\frac{1}{x} - \frac{1}{x^2}}{2 + 5\frac{1}{x^2}} = \frac{1}{2}.$$

6.

If $x < 1$, then $|x - 1| = -(x - 1)$, so

$$\lim_{x\to 1^-} \frac{|x - 1|}{x - 1} = \lim_{x\to 1^-} \frac{-(x - 1)}{x - 1} = -1.$$

7. We have

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (x^3 - \sin x) = 0^3 - \sin 0 = 0$$

and

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (e^x - 1) = e^0 - 1 = 0.$$

We know that x^3, $\sin x$, and e^x are continuous everywhere, so $f(x)$ is continuous away from 0. The above computation shows that $\lim_{x\to 0} f(x) = 0$, and $f(0) = 0$, so $f(x)$ is also continuous at 0.

8. A function f on a domain D is continuous at a point $a \in D$ if, for all $\epsilon > 0$, there exists a $\delta > 0$ such that if $x \in D$ and $|x - a| < \delta$, then $|f(x) - f(a)| < \epsilon$.

9. A function f on a domain D is continuous at a point $a \in D$ if and only if, for all sequences $\{x_n\}$ of real numbers such that $x_n \in D$ for all n and $\lim_{n\to\infty} x_n = a$, we have

$$\lim_{n\to\infty} f(x_n) = f(a).$$

10. A function f on a domain D is continuous at a point $a \in D$ if and only if

$$\lim_{x\to a} f(x) = f(a).$$

11. Consider $f(x) = 1/x$ on the interval $(0, 1)$.

12. Sketch: suppose that the leading term of f is positive. Then, we have that $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to-\infty} f(x) = -\infty$. In particular, $f(x) > 0$ for some x and $f(x) < 0$ for some x. By the intermediate value theorem, there exists an x such that $f(x) = 0$. If the leading term of f is negative, then $\lim_{x\to\infty} f(x) = -\infty$ and $\lim_{x\to-\infty} f(x) = \infty$. As before, we conclude by the intermediate value theorem that f has a root.

13. Consider the function

$$d(x) = \sqrt{(x - x_0)^2 + (f(x) - y_0)^2}.$$

Then $d(x)$ is the distance between $(x, f(x))$ and (x_0, y_0). The function $d(x)$ is continuous, so has a minimum on the interval $[a, b]$.

14. We have

$$\lim_{x\to 0^-} \frac{f(x)}{x} = \lim_{x\to 0^-} \frac{-x^3}{x} = 0$$

and

$$\lim_{x\to 0^+} \frac{f(x)}{x} = \lim_{x\to 0^+} \frac{x^3}{x} = 0.$$

15. For $x > 0$, we have $f(x) = x^3$, and for $x < 0$, we have $f(x) = -x^3$. Thus, $f(x)$ is certainly differentiable away from 0. At $a = 0$, we showed above that $f'(0)$ exists, and is equal to 0, so f is also differentiable at 0.

16. The derivative of $f(x)$ is given by

$$f'(x) = \begin{cases} 3x^2 & \text{if } x \geq 0 \\ -3x^2 & \text{if } x < 0 \end{cases}$$

The second derivative exists, and is

$$f''(x) = \begin{cases} 6x & \text{if } x \geq 0 \\ -6x & \text{if } x < 0 \end{cases}$$

That is, $f''(x) = 6|x|$. This is not differentiable at 0. Thus, f is twice differentiable, but not thrice differentiable.