Review for Midterm 3 (Math 3210, Fall 2023)

1 Sample Problems

1. Prove directly from the definition that the function $f(x) = 2|x| - 1$ is continuous at every point of $\mathbb{R}$.
2. Prove that the function
$$f(x) = \begin{cases} x^2 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$
is continuous at every point of $\mathbb{R}$.
3. Consider the function
$$f(x) = \begin{cases} x \sin 1/x & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$
Compute
$$\lim_{x \to 0} f(x)$$
and prove your answer is correct.
4. If $f(x)$ is the function in the previous problem, prove that f is continuous on all of $\mathbb{R}$ (you can take for granted that $\sin x$ is continuous).
5. Compute
$$\lim_{x \to \infty} \frac{x^2 - 3x - 1}{2x^2 + 5}$$
and prove that your answer is correct.
6. Compute
$$\lim_{x \to 1^-} \frac{|x - 1|}{x - 1}$$
7. Consider the function
$$f(x) = \begin{cases} x^3 - \sin x & \text{if } x \geq 0 \\ e^x - 1 & \text{if } x < 0. \end{cases}$$
Compute $\lim_{x \to 0^-} f(x)$ and $\lim_{x \to 0^+} f(x)$. Prove that $f(x)$ is continuous at every point of $\mathbb{R}$.
8. State the definition of continuity of a function at a point.
9. State the characterization of continuity in terms of limits of sequences.
10. State the characterization of continuity in terms of limits of functions.
11. Find an example of a function which is continuous on a bounded interval but does not have a maximum value.

12. Prove that every polynomial of odd degree has a real root. **Hint**: use the intermediate value theorem.

13. Prove that if f is a continuous function on a closed bounded interval $[a, b]$ and if (x_0, y_0) is any point in the plane, then there is a point on the graph of f which is closest to (x_0, y_0).

14. Consider the function $f(x) = |x|^3$. Compute

$$\lim_{x \to 0^-} \frac{f(x)}{x} \quad \text{and} \quad \lim_{x \to 0^+} \frac{f(x)}{x}.$$

15. If $f(x)$ is the function in the previous problem, prove that $f(x)$ is differentiable at every point of $\mathbb{R}$.

16. Is the function $f(x)$ in the previous problem twice differentiable? What about thrice?