Midterm 2 Math 3210 Fall 2023

Directions:

- You have 50 minutes to complete this exam.
- You may cite results proved during lecture or in the book without repeating the proof.
- If you wish to use a result from lecture or from the book, you must write out the complete *statement* of the result which you are citing.

Question	Points	Score
1	10	
2	10	
3	10	
Total:	30	

1. (10 points) Prove that

$$\lim_{n\to\infty} \frac{2n+1}{n+3} = 2$$

directly from the definition of a limit (that is, **without** citing any theorems from lecture or the book!).

Solution: Fix $\varepsilon > 0$. Set $N = \frac{5}{\varepsilon}$. If $n > N$, then this implies that we have

$$\frac{5}{n} < \varepsilon,$$

and therefore

$$\left|\frac{2n+1}{n+3} - 2\right| = \left|\frac{-5}{n+3}\right| = \frac{5}{n+3} < \frac{5}{n} < \varepsilon.$$

Therefore the limit is 2.

2. (10 points) Let $\{a_n\}$ be a sequence of real numbers. Show that if the sequence $\{na_n\}$ converges then $\lim_{n\to\infty} a_n = 0$.

Solution 1: As the sequence $\{na_n\}$ converges, it must be bounded. Thus, there exists a positive constant $C \in \mathbb{R}$ such that

$$|na_n| < C$$

for all n. Dividing by n, this implies that

$$|a_n| < \frac{C}{n}$$

for all n. Now fix $\varepsilon > 0$. Let $N = \frac{C}{\varepsilon}$. If $n > N$, then we have

$$\frac{C}{n} < \varepsilon,$$

and therefore

$$|a_n| < \frac{C}{n} < \varepsilon.$$

Thus, the sequence $\{a_n\}$ converges to 0.

Solution 2: Consider the sequences $\left\{\frac{1}{n}\right\}$ and $\{na_n\}$. We have

$$\lim_{n\to\infty} \frac{1}{n} = 0$$

and

$$\lim_{n\to\infty} na_n = a$$

for some $a \in \mathbb{R}$. By the main limit theorem, we get that

$$\lim_{n\to\infty} \left(\frac{1}{n} \cdot na_n\right) = 0 \cdot a.$$

Therefore, we have

$$\lim_{n\to\infty} a_n = 0.$$

3. (10 points) Define a sequence $\{a_n\}$ of real numbers by $a_1 = 1$ and $a_{n+1} = \frac{2a_n+1}{5}$. The first four terms of this sequence are

$$a_1, a_2, a_3, a_4, \ldots = 1, \frac{3}{5}, \frac{11}{25}, \frac{47}{125}, \ldots$$

(a) Prove that $\{a_n\}$ is decreasing and is bounded below by 0.

Solution: It will suffice to show that

$$0 \leq a_{n+1} \leq a_n$$

for all $n \in \mathbb{N}$. We will prove this by induction on n. The base case of $n = 1$ is the inequalities

$$0 \leq \frac{3}{5} \leq 1$$

which are true. For the induction step, suppose that we already know that

$$0 \leq a_{n+1} \leq a_n$$

is true for some n. We multiply by 2, add 1, then divide by 5 to get

$$\frac{1}{5} \leq \frac{2a_{n+1}+1}{5} \leq \frac{2a_n+1}{5}.$$

Applying the recursion relation defining our sequence, this is the same thing as

$$\frac{1}{5} \leq a_{n+2} \leq a_{n+1}.$$

As $\frac{1}{5} > 0$, this implies the inequalities hold also for $n+1$. By induction, they hold for all n.

(b) Prove that $\{a_n\}$ converges.

Solution: In part **(a)**, we showed that the sequence $\{a_n\}$ is decreasing and bounded below. As it is decreasing, it is also bounded above (by a_1). Thus the sequence is monotone and bounded. By the monotone convergence theorem, the sequence converges.

(c) Find $\lim_{n\to\infty} a_n$.

Solution: We take the limit of both sides of the recursion relation to get

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{2a_n + 1}{5}.$$

Applying the main limit theorem, this becomes

$$\lim_{n\to\infty} a_{n+1} = \frac{2(\lim_{n\to\infty} a_n) + 1}{5}.$$

Say that $\lim_{n\to\infty} a_n = a$. Then also $\lim_{n\to\infty} a_{n+1} = a$, so the above implies that

$$a = \frac{2a + 1}{5}.$$

Solving for a, we get

$$a = \frac{1}{3}.$$

Definitions and theorems:

Definition 1 *A sequence $\{a_n\}$ of real numbers* ***converges*** *to a real number a if for all $\varepsilon > 0$ there exists a real number N such that for all natural numbers $n > N$, we have*

$$|a_n - a| < \varepsilon.$$

Definition 2 *A sequence $\{a_n\}$ of real numbers is* ***decreasing*** *if $a_{n+1} \leq a_n$ for all n, is* ***increasing*** *if $a_{n+1} \geq a_n$ for all n, and is* ***monotonic*** *if it is either increasing or decreasing.*

Theorem 1 (The Monotone Convergence Theorem) *If $\{a_n\}$ is a sequence of real numbers which is both bounded and monotonic, then $\{a_n\}$ converges.*