Review for Midterm 2 (Math 3210, Fall 2023)

1 Solutions to sample problems

1. Apply the triangle inequality $|x+y| \leq |x| + |y|$ with $x = a-b$ and $y = b$ to get

$$|a| = |(a-b)+b| \leq |a-b| + |b|.$$

Then subtract $|b|$ from both sides to get

$$|a| - |b| \leq |a-b|.$$

2. Fix $\varepsilon > 0$. As $a_n \to a$, there exists N such that $|a_n - a| < \varepsilon$ for all $n > N$. Thus, if $n > N$, we have that $n+1 > N$, so $|a_{n+1} - a| < \varepsilon$, and therefore $|b_n - a| < \varepsilon$.

3. Fix $\varepsilon > 0$. Set $N = \frac{1}{\varepsilon^{2/3}}$. If $n > N$, then $\frac{1}{n^{3/2}} < \varepsilon$, so

$$\left| \frac{1}{n\sqrt{n}} - 0 \right| < \varepsilon.$$

4. We note that $3 + 2/n > 3$, so we have $\frac{1}{3+2/n} < \frac{1}{3}$. We compute that

$$\left| \frac{n}{2n+2} - \frac{1}{3} \right| = \frac{1}{3} \cdot \frac{2/n}{3+2/n} < \frac{2}{9n}.$$

Fix $\varepsilon > 0$. Set $N = \frac{2}{9\varepsilon}$. If $n > N$, then $\frac{2}{9n} < \varepsilon$, and therefore

$$\left| \frac{n}{2n+2} - \frac{1}{3} \right| < \frac{2}{9n} < \varepsilon.$$

5. We compute that

$$\left| \frac{\sqrt{9n^2+n}}{2n} - \frac{3}{2} \right| = \frac{1}{2} \cdot \frac{\frac{1}{n}}{\sqrt{9+\frac{1}{n}}+3}$$

We note that $9 + \frac{1}{n} > 9$, so $\sqrt{9+\frac{1}{n}} > 3$. This implies $\sqrt{9+\frac{1}{n}} + 3 > 6$, which implies $\frac{1}{\sqrt{9+\frac{1}{n}}+3} < \frac{1}{6}$. Thus, we obtain

$$\left| \frac{\sqrt{9n^2+n}}{2n} - \frac{3}{2} \right| = \frac{1}{2} \cdot \frac{\frac{1}{n}}{\sqrt{9+\frac{1}{n}}+3} < \frac{1}{12n}.$$

Now fix $\varepsilon > 0$. Set $N = \frac{1}{12\varepsilon}$. If $n > N$, then $\frac{1}{12n} < \varepsilon$, and therefore by the above computation we have

$$\left| \frac{\sqrt{9n^2+n}}{2n} - \frac{3}{2} \right| < \frac{1}{12n} < \varepsilon.$$

6. We compute that

$$\left| \frac{4n+(-1)^n}{2n+5} - 2 \right| = \left| \frac{(-1)^n - 10}{2n+5} \right| = \frac{10-(-1)^n}{2n+5} \leq \frac{11}{2n+5}.$$

Fix $\varepsilon > 0$. Set $N = \frac{1}{2}\left(\frac{11}{\varepsilon} - 5\right)$. If $n > N$, then $\frac{11}{2n+5} < \varepsilon$, so

$$\left| \frac{4n+(-1)^n}{2n+5} - 2 \right| \leq \frac{11}{2n+5} < \varepsilon.$$

7. We know that $\lim \frac{1}{n^k} = 0$ for any $k \geq 1$. By the main limit theorem, we have

$$\lim_{n\to\infty} \frac{3n^4+2n^3+n-1}{5n^4+2n^3+2} = \lim_{n\to\infty} \frac{3+2\frac{1}{n}+\frac{1}{n^3}-\frac{1}{n^4}}{5+2\frac{1}{n}+2\frac{1}{n^4}} = \frac{3+2\lim\frac{1}{n}+\lim\frac{1}{n^3}-\lim\frac{1}{n^4}}{5+2\lim\frac{1}{n}+2\lim\frac{1}{n^4}} = \frac{3+0+0-0}{5+0+0} = \frac{3}{5}.$$

8. We compute that

$$\sqrt{n^4+2n^2+1} - n^2 = \frac{2n^2+1}{\sqrt{n^4+2n^2+1}+n^2} = \frac{2+\frac{1}{n^2}}{\sqrt{1+2\frac{1}{n^2}+\frac{1}{n^4}}+1}.$$

Applying the main limit theorem, we get

$$\lim_{n\to\infty}\left(\sqrt{n^4+2n^2+1}-n^2\right) = \frac{\lim\left(2+\frac{1}{n^2}\right)}{\lim\left(\sqrt{1+2\frac{1}{n^2}+\frac{1}{n^4}}+1\right)} = \frac{2+\lim\frac{1}{n^2}}{\sqrt{1+\lim 2\frac{1}{n^2}+\lim\frac{1}{n^4}}+1} = \frac{2+0}{\sqrt{1+0+0}+1} = 1.$$

9. We will first show that if $n \geq 10$ then $n^3 < 2^n$. We prove this by induction. The base case is $n = 10$, which is the inequality $10^3 < 2^{10}$, or in other words $1000 < 1024$, which is true. Suppose that we have shown that $n^3 < 2^n$ for some n. It follows that $2n^3 < 2^{n+1}$. We claim that $(n+1)^3 < 2n^3$. Indeed, rearranging this, this is equivalent to $n^3 - 3n^2 - 3n - 1 > 0$. Factoring this gives $n(n(n-3)-1)-1 > 0$. As $n \geq 10$, we have $n(n-3) > 1$, so this is definitely true. We get

$$(n+1)^3 < 2n^3 < 2^{n+1},$$

so the result is true by induction.

We now return to the problem at hand. Fix $M \in \mathbb{R}$. Set $N = \max\{M, 10\}$. If $n > N$, then using the above inequality we get

$$\frac{2^n}{n^2} > \frac{n^3}{n^2} = n > N > M.$$

Therefore $\lim \frac{2^n}{n^2} = \infty$.

10. (solution omitted)

11. We will show that the sequence $\{a_n\}$ is increasing and bounded above by 2. To prove this, we will show that

$$a_n < a_{n+1} < 2.$$

We use induction on n. We have $a_1 = 0$ and $a_2 = \frac{3}{2}$, and $1 < \frac{3}{2} < 2$, so this is true for $n = 1$. For the induction step, assume that

$$a_n < a_{n+1} < 2$$

for some n. We divide by 2 and add 1 to get

$$\frac{1}{2}a_n + 1 < \frac{1}{2}a_{n+1} + 1 < \frac{1}{2} \cdot 2 + 1 = 2.$$

Using the recursive definition of the sequence $\{a_n\}$, this is equivalent to

$$a_{n+1} < a_{n+2} < 2.$$

The result follows by induction.

We have shown that $\{a_n\}$ is increasing (hence monotone) and bounded. By the monotone convergence theorem, $\lim a_n$ exists.

12. We know that the limit $\lim a_n$ exists (and is finite). Say $a = \lim a_n$. We take the limit of both sides of the recursion relation to get

$$\lim a_{n+1} = \lim \left(\frac{1}{2}a_n + 1 \right).$$

By the main limit theorem, we get

$$\lim a_{n+1} = \frac{1}{2} \lim a_n + 1.$$

By question **1.**, we have

$$\lim a_{n+1} = \lim a_n = a.$$

Thus, we get

$$a = \frac{1}{2}a + 1.$$

Solving for a gives $a = 2$.

13. We claim that

$$a_n = \frac{2^{n-1} - 1}{2^{n-2}}$$

for all n. We prove this by induction on n. The base case of $n = 1$ is true. For the induction step, suppose that we know

$$a_n = \frac{2^{n-1} - 1}{2^{n-2}}$$

for some n. This implies

$$a_{n+1} = \frac{1}{2}a_n + 1 = \frac{1}{2} \cdot \frac{2^{n-1} - 1}{2^{n-2}} = \frac{2^n - 1}{2^{n-1}}.$$

The result follows by induction.

By the way, once we know this formula, we get another way of seeing that $\lim a_n = 2$. Indeed, using the main limit theorem and the fact that $\lim \frac{1}{2^n} = 0$, we get

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2^{n-1}-1}{2^{n-2}} = \lim_{n\to\infty} \frac{\frac{1}{2}-\frac{1}{2^n}}{\frac{1}{2^2}} = \frac{\frac{1}{2}-0}{\frac{1}{2^2}} = 2.$$

14.

- $a_n = (-1)^n$.
- $a_n = (-1)^n n$.
- $a_n = n$.
- $a_n = n + (-1)^n$.

15.

- $a_n = \frac{1}{n}$, $b_n = 3n$.
- $a_n = n$, $b_n = 5 - n$.