Review for Midterm 2 (Math 3210, Fall 2023)

1 Sample Problems

1. Prove that if a and b are real numbers then $|a| - |b| \leq |a - b|$ (**Hint:** use the triangle inequality).

2. Let $\{a_n\}$ be a sequence of real numbers. Define a new sequence $\{b_n\}$ by $b_n = a_{n+1}$. Show that if $a_n \to a$ for some real number a, then also $b_n \to a$.

3. Prove directly from the definition of a limit (ie. without using the main limit theorem) that
$$\lim_{n\to\infty} \frac{1}{n\sqrt{n}} = 0.$$

4. Prove directly from the definition of a limit (ie. without using the main limit theorem) that
$$\lim_{n\to\infty} \frac{n}{3n+2} = \frac{1}{3}.$$

5. Prove directly from the definition of a limit (ie. without using the main limit theorem) that
$$\lim_{n\to\infty} \frac{\sqrt{9n^2+n}}{2n} = \frac{3}{2}.$$

6. Prove directly from the definition of a limit (ie. without using the main limit theorem) that
$$\lim_{n\to\infty} \frac{4n+(-1)^n}{2n+5} = 2.$$

7. Compute
$$\lim_{n\to\infty} \frac{3n^4+2n^3+n-1}{5n^4+2n^3+2}$$
and prove that your answer is correct (using whatever tools you like).

8. Compute
$$\lim_{n\to\infty} \left(\sqrt{n^4+2n^2+1} - n^2\right)$$
and prove that your answer is correct (using whatever tools you like).

9. Prove that
$$\lim_{n\to\infty} \frac{2^n}{n^2} = \infty.$$

10. (Challenge problem! Only try this if you feel good about all of the rest of the problems). Generalize the previous problem by proving that for any natural number $k \in \mathbb{N}$, we have
$$\lim_{n\to\infty} \frac{2^n}{n^k} = \infty.$$

11. Define a sequence $\{a_n\}$ of real numbers recursively by

$$a_1 = 0 \qquad \text{and} \qquad a_{n+1} = \frac{1}{2}a_n + 1.$$

Using the monotone convergence theorem, prove that $\lim a_n$ exists.

12. For the sequence $\{a_n\}$ in the previous problem, compute $\lim a_n$ explicitly using the method from lecture.

13. For the sequence $\{a_n\}$ in the previous problem, find a general formula for a_n in terms of n (**Hint:** to guess the formula, try writing out the first few terms. Then prove your formula is correct by induction).

14. For each of the following, find an example of a sequence $\{a_n\}$ with the given properties:

- $\{a_n\}$ is bounded and $\lim_{n\to\infty} a_n$ does not exist.
- $\{a_n\}$ is not bounded and $\lim_{n\to\infty} a_n$ does not exist.
- $\{a_n\}$ is not bounded and $\lim_{n\to\infty} a_n$ exists.
- $\{a_n\}$ is not monotone and $\lim_{n\to\infty} a_n = \infty$.

15. For each of the following, find an example of a pair of sequences $\{a_n\}$ and $\{b_n\}$ with the given properties:

- $\lim_{n\to\infty} a_n = 0$ and $\lim_{n\to\infty} a_n b_n = 3$.
- $\lim_{n\to\infty} a_n = \infty$ and $\lim_{n\to\infty}(a_n + b_n) = 5$.