Midterm 1 Math 3210 Fall 2023

Directions:

- You have 50 minutes to complete this exam.
- You may cite results proved during lecture or in the book without repeating the proof.
- If you wish to use a result from lecture or from the book, you must write out the complete *statement* of the result which you are citing.

Question	Points	Score
1	10	
2	10	
3	10	
Total:	30	

1. (10 points) For each of the following subsets of $\mathbb{R}$, find the supremum and the infimum. You do **not** have to justify your answers at all.

   (a) $(-4, 3] \cup (5, 7]$

   Solution:
   $$\sup = 7 \qquad \inf = -4$$

   (b) $\{x \in \mathbb{R} | x^2 < 1\}$

   Solution: The set is equal to $(-1, 1)$, so the answer is
   $$\sup = 1 \qquad \inf = -1.$$

   (c) $\left\{ \frac{15n-1}{3n+1} | n \in \mathbb{N} \right\}$

   Solution: For this problem I found it useful to note that
   $$\frac{15n-1}{3n+1} = 5 + \frac{-6}{3n+1}.$$
   The answer is
   $$\sup = 5 \qquad \inf = \frac{7}{2}.$$

2. (10 points) Let $I \subset \mathbb{R}$ and $J \subset \mathbb{R}$ be Dedekind cuts. Prove that the set $L = I \cup J$ is also a Dedekind cut.

(Definition of a Dedekind cut): A subset $L \subset \mathbb{Q}$ is a *Dedekind cut* if

(1) $L \neq \emptyset$ and $L \neq \mathbb{Q}$,

(2) For any element $r \in L$, there exists an element $s \in L$ such that $r < s$, and

(3) If s is a rational number such that $s < r$ for some $r \in L$, then $s \in L$.

Solution: We will check that the three conditions in the definition of a Dedekind cut hold for L.

(1): As I and J are Dedekind cuts, they each satisfy **(1)**, so $I \neq \emptyset$ and $J \neq \emptyset$. It follows that $I \cup J \neq \emptyset$. Furthermore, we have that $I \neq \mathbb{Q}$, so we can find $x \in \mathbb{Q}$ such that $x \notin I$. As I satisfies **(3)**, we have that x is an upper bound for I. Similarly, we can find $y \in \mathbb{Q}$ such that $y \notin J$, and hence y is an upper bound for J. Let z be a rational number such that $z \geq x$ and $z \geq y$. Then $z \notin I$ and $z \notin J$, so $z \notin I \cup J$, and hence $L \neq \mathbb{Q}$.

(2): Let r be an element of L. Then either $r \in I$ or $r \in J$ (or both). Suppose without loss of generality that $r \in I$. As I is a Dedekind cut, it satisfies **(2)**, so there exists $s \in I$ such that $r < s$. Then s is an element of L which satisfies $r < s$.

(3): Let r and s be rational numbers such that $s < r$ and such that $r \in L$. Then either $r \in I$ or $r \in J$ (or both). Suppose without loss of generality that $r \in I$. As I is a Dedekind cut, it satisfies **(3)**, so $s \in I$, and therefore $s \in L$.

3. (10 points) Let x be a real number. Prove that the supremum of the set $\{r \in \mathbb{Q} | r < x\}$ is equal to x. **Hint:** you might want to use problem 7 from Homework 3, which says that if x and y are real numbers such that $x < y$, then there is a rational number r such that $x < r < y$.

Solution: Say that the set is A. For any $r \in A$, we have $r < x$ (by definition of A!). Therefore x is an upper bound for A. We will show that x is the least upper bound. Suppose that y is a real number such that $y < x$. By problem 7 from Homeword 3, there exists a rational number r such that $y < r < x$. This r is an element of A which is strictly greater than y. Therefore y is not an upper bound for A. We have shown that if y is a real number such that $y < x$, then y is not an upper bound for A. It follows that x is the least upper bound for A.