Review for Midterm 1 (Math 3210, Fall 2023)

1 Solutions to sample problems

1. Base case: we check the result directly for $n = 1$ ($1^2 < 3$) and $n = 2$ ($2^2 < 3^2$). Induction step: suppose that $n^2 < 3^n$ for some $n \geq 2$. Multiply both sides by 3 to get

$$3n^2 < 3^{n+1}.$$

We claim that $(n+1)^2 \leq 3n^2$. Indeed, after rearranging and factoring, this is equivalent to the inequality

$$2n(n-1) - 1 \geq 0$$

which is true because we have $n(n-1) \geq 1$ as $n \geq 2$. Combining our two inequalities, we get

$$(n+1)^2 \leq 3n^2 < 3^{n+1}$$

which shows that the result holds true for $n+1$ as well. The result is therefore true by induction.

2. We take our base case to be $n = 4$. Then the claim is that $2^4 < 4!$, or equivalently $16 < 24$. For the induction step, suppose that $2^n < n!$ for some $n \geq 4$. Using the inequality $2 \leq n+1$, we get

$$2^{n+1} = 2 \cdot 2^n \leq (n+1)2^n < (n+1) \cdot n! = (n+1)!$$

Thus the inequality holds for $n+1$ as well. The result follows for all $n \geq 4$ by induction.

3. Base case: $\frac{1}{1(1+1)} = \frac{1}{1+1}$. Induction step: assume that we know

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}$$

for some $n \geq 1$. Add $\frac{1}{(n+1)(n+2)}$ to both sides to get

$$\sum_{k=1}^{n+1} \frac{1}{k(k+1)} = \frac{1}{(n+1)(n+2)} + \sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{1}{(n+1)(n+2)} + \frac{n}{n+1} = \frac{n+1}{n+2}.$$

Thus the inequality holds for $n+1$ as well. The result follows for all $n \geq 1$ by induction.

4. Base case: $7^2 - 1 = 48$ is divisible by 48. Induction step: suppose that we know that $7^{2n} - 1$ is divisible by 48 for some $n \geq 1$. We have

$$7^{2n+2} - 1 = 7^2 \cdot 7^{2n} - 1 = (48+1) \cdot 7^{2n} - 1 = 48 \cdot 7^{2n} + (7^{2n} - 1).$$

Both terms on the right hand side are divisible by 48, so $7^{2n+2} - 1$ is divisible by 48. By induction, the result is true for all $n \geq 1$.

5. By **(M4.)**, there exists an element $z^{-1} \in F$ such that $z \cdot z^{-1} = 1$. We multiply both sides of $xz = yz$ by z^{-1} on the right to get

$$(xz)z^{-1} = (yz)z^{-1}.$$

By **(M2.)**, this implies

$$x(zz^{-1}) = y(zz^{-1})$$

hence

$$x \cdot 1 = y \cdot 1$$

and so $x = y$.

For the second part, suppose that $xy = 0$. The result we are trying to show is equivalent to the statement that if $x \neq 0$, then $y = 0$. By **(M3.)**, we have $xy = yx$, so $yx = 0$. Therefore $yx = 0 = 0 \cdot x = x \cdot 0$. We now apply the previous part to the equation

$$y \cdot x = 0 \cdot x$$

and use the assumption that $x \neq 0$ to conclude that $y = 0$.

6. We need to verify the three axioms in the definition of a Dedekind cut. For the first, we note that $0 \in J$, so $J \neq \emptyset$, and $100 \notin J$, so $J \neq \mathbb{Q}$. For the second, suppose that $r \in J$. We will show that there exists $s \in J$ such that $r < s$. As $0 \in J$, this is certainly true if $r < 0$. Suppose that $r \geq 0$. We claim that there exists a positive rational number a such that $3r^2a + 3ra^2 + a^3 < 5 - r^3$. Indeed, suppose that

$$0 < a < \frac{5 - r^3}{3r^2 + 3r + 1}$$

and that $0 < a < 1$ (note that the RHS is > 0). Then $a^2 < a$, so

$$3r^2a + 3ra^2 + a^3 < 3r^2a + 3ra + a = (3r^2 + 3r + 1)a < 5 - r^3.$$

Now, we have

$$(r + a)^3 = r^3 + (3r^2a + 3ra^2 + a^3) < r^3 + (5 - r^3) = 5.$$

Therefore $r + a \in J$, and as $a > 0$ we have $r < r + a$. Finally, we need to show that J is downward closed. Suppose that $r \in J$ and s is a rational number such that $s < r$. If $s \leq 0$, then $s^3 \leq 0$, so $s \in J$. Suppose that $s > 0$. It follows that $r > 0$. We obtain

$$s^3 = s \cdot s^2 < r \cdot s^2 < r \cdot r^2 = r^3 < 5.$$

Thus $s \in J$.

7. Let s be a rational number and assume for the sake of contradiction that $J = L_s$. We know that s is the least upper bound of L_s, so s is the least upper bound of J. We have shown that J is a Dedekind cut, so J does not have a largest element. Thus $s \notin J$, and so $s^3 \geq 5$. On the other hand, as s is rational, we have $s^3 \neq 5$, and therefore $s^3 > 5$. As in the previous question, we can find a positive rational number $a > 0$ such that

$$(s - a)^3 = s^3 - (3s^2a - 3sa^2 + a^3) > s^3 - (s^3 - 5) = 5$$

Then $s - a$ is an element of L_s which is not in J, a contradiction.

8.

$$\sup A = b$$
$$\inf A = -\infty$$

$$\sup B = \infty$$
$$\inf B = -\infty$$

$$\sup C = b$$
$$\inf C = a$$

$$\sup D = 1$$
$$\inf D = 0$$

$$\sup E = \infty$$
$$\inf E = 0$$

$$\sup F = \sqrt{2}$$
$$\inf F = -\sqrt{2}$$