1 The Main Limit Theorem

The goal of this note is to give a complete proof of Taylor's "Main Limit Theorem". We will first prove some lemmas.

Lemma 1.1 (The triangle inequality)**.** *If a, b are two real numbers, then*

$$|a+b| \leq |a| + |b|.$$

Proof. See Taylor, Theorem 2.1.2. $\square$

Lemma 1.2. *Let $\{a_n\}$ be a sequence of real numbers and let a be a real number. Then $a_n \to a$ if and only if $a_n - a \to 0$.*

Proof. We will prove this by just writing down the definition of "convergence" in the two cases, and then observing that we've written down the same thing twice. Indeed, the statement that $a_n \to a$ means that for any $\varepsilon > 0$, there exists N such that for all $n > N$ we have $|a_n - a| < \varepsilon$. The statement that $a_n - a \to 0$ is the means that for any $\varepsilon > 0$, there exists N such that for all $n > N$ we have $|(a_n - a) - 0| < \varepsilon$. $\square$

Lemma 1.3. *Let $\{a_n\}$ and $\{b_n\}$ be two sequences of real numbers. If $a_n \to 0$ and $\{b_n\}$ is bounded, then $a_n b_n \to 0$.*

Proof. Say that C is a real number such that $|b_n| \leq C$ for all n. We may assume that $C > 0$. Fix $\varepsilon > 0$. As $a_n \to a$, there exists an N such that for all $n > N$ we have

$$|a_n - 0| < \frac{\varepsilon}{C}.$$

It follows that

$$|a_n b_n| < \frac{\varepsilon}{C} C = \varepsilon.$$

We conclude that $a_n b_n \to 0$, as claimed. $\square$

Example 1.4. The assumption in Lemma 1.3 that the sequence $\{b_n\}$ is bounded is important. For instance, consider $a_n = \frac{1}{n}$ and $b_n = n$. Then $a_n \to 0$, but $a_n b_n = 1$ for all n, so $a_n b_n \to 1$. So, $a_n b_n$ converges to 1, not to 0. We can even make the 1 into any other real number! Indeed, taking $a_n = \frac{1}{n}$ and $b_n = kn$ (for some $k \in \mathbb{R}$), we get that $a_n b_n = k$, so $a_n b_n \to k$.

For an even worse example, consider $a_n = \frac{1}{n}$ and $b_n = n^2$. Then we have $a_n b_n = n$, so $a_n b_n$ does not converge at all!

Lemma 1.5. *Let $\{a_n\}$ be a sequence of real numbers, let a be a real number, and suppose that $a_n \to a$.*

(1) $\{a_n\}$ is bounded.

(2) If $a_n \neq 0$ for all $n \in \mathbb{N}$ and $a \neq 0$, then $\left\{\frac{1}{a_n}\right\}$ is bounded.

Proof. For the first part, choose a number $\varepsilon > 0$. We know that there exists an N such that if $n > N$ then $|a_n - a| < \varepsilon$. It follows that

$$|a_n| \leq |a_n - a| + |a| < \varepsilon + a.$$

This gives a bound for $|a_n|$ for all $n > N$. Thus, if we pick a constant C which is greater than $\varepsilon + a$, and is also greater than $|a_n|$ for each of the finitely many n such that $n \leq N$, then we will have $|a_n| < C$ for all $n \in \mathbb{N}$. This shows that $\{a_n\}$ is bounded.

For the second part, choose a number $\varepsilon > 0$ such that $0 < \varepsilon < |a|$. Thus, we have

$$0 < |a| - \varepsilon \tag{1.5.1}$$

We know that there exists an N such that if $n > N$ then $|a_n - a| < \varepsilon$. Negating this, we get

$$-\varepsilon < -|a_n - a|. \tag{1.5.2}$$

Applying the triangle inequality, we get

$$|a| = |(a - a_n) + a_n| \le |a - a_n| + |a_n|,$$

and therefore by subtraction

$$|a| - |a_n - a| \le |a_n| \tag{1.5.3}$$

(here I also used that $|a - a_n| = |a_n - a|$). Combining (1.5.1), (1.5.2), and (1.5.3), we get

$$0 < |a| - \varepsilon < |a| - |a_n - a| \le |a_n|.$$

It follows that

$$\frac{1}{|a_n|} < \frac{1}{|a| - \varepsilon}$$

for all $n > N$. As in the previous part, we can get a bound which works for all n by taking a number larger than this bound and which is also larger than $1/|a_n|$ for all $n \le N$. $\square$

We are now ready to prove the Main Limit Theorem.

Theorem 1.6 (Main Limit Theorem). *Let $\{a_n\}$ and $\{b_n\}$ be two sequences of real numbers, let $a, b \in \mathbb{R}$, and suppose that $a_n \to a$ and $b_n \to b$.*

(a) $ca_n \to ca$ for any $c \in \mathbb{R}$.

(b) $a_n + b_n \to a + b$.

(c) $a_n b_n \to ab$.

(d) $\frac{a_n}{b_n} \to \frac{a}{b}$, if $b_n \neq 0$ for all n and $b \neq 0$.

(e) $a_n^k \to a^k$ for any $k \in \mathbb{N}$.

(f) $a_n^{1/k} \to a^{1/k}$ for any $k \in \mathbb{N}$, if $a_n \ge 0$ for all n.

Proof. **(a):** By Lemma 1.2, we have $a_n - a \to 0$. The constant sequence $\{c\}$ is bounded, so we can apply Lemma 1.3 to the product $c(a_n - a)$ to conclude that $c(a_n - a) \to 0$. Thus $ca_n - ca \to 0$, so by Lemma 1.2 we get $ca_n \to ca$.

(b): Fix $\varepsilon > 0$. As $a_n \to a$ and $b_n \to b$, we can find constants N_1, N_2 such that

$$|a_n - a| < \frac{\varepsilon}{2}$$

for $n > N_1$ and

$$|b_n - b| < \frac{\varepsilon}{2}$$

for $n > N_2$. Now if $N > \max(N_1, N_2)$, then

$$|(a_n + b_n) - (a + b)| = |(a_n - a) + (b_n - b)| \le |a_n - a| + |b_n - b| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Here, the $\leq$ step is the triangle inequality.

(c): We note that

$$a_n b_n - ab = a_n b_n - a_n b + a_n b - ab = a_n(b_n - b) + (a_n - a)b.$$

By Lemma 1.2, we have $b_n - b \to 0$. By Lemma 1.5, the sequence $\{a_n\}$ is bounded. By Lemma 1.3, we get that $a_n(b_n - b) \to 0$. Similarly, we have $a_n - a \to 0$, so by part **(a)**, we have $(a_n - a)b \to 0$. Applying part **(b)**, we conclude that $a_n(b_n - b) + (a_n - a)b \to 0$. Thus, $a_n b_n - ab \to 0$, and therefore $a_n b_n \to ab$.

(d): We note that

$$\frac{a_n}{b_n} - \frac{a}{b} = \frac{1}{b_n b}(a_n b - a b_n).$$

By part **(a)**, we have $a_n b \to ab$ and $-ab_n \to -ab$. By part **(b)**, we have that $a_n b - ab_n \to 0$. By part **(a)**, we have $b_n b \to b^2$, which is nonzero by assumption, so by Lemma 1.5, the sequence $\left\{\frac{1}{b_n b}\right\}$ is bounded. Thus, by Lemma 1.3, we get that $\frac{1}{b_n b}(a_n b - ab_n) \to 0$. Thus $\frac{a_n}{b_n} - \frac{a}{b} \to 0$, and therefore $\frac{a_n}{b_n} \to \frac{a}{b}$.

(e): We induct on k. The case $k = 1$ is true by assumption. Suppose that for some k we have $a_n^k \to a^k$. By part **(c)**, we have that $a_n \cdot a_n^k \to a \cdot a^k$. Thus, $a_n^{k+1} \to a^{k+1}$. The result follows by induction.

(f): We consider first the case when $a \neq 0$. We use the identity

$$x^k - y^k = (x - y)(x^{k-1} + x^{k-2}y + \ldots + xy^{k-2} + y^{k-1})$$

which holds for any $x, y \in \mathbb{R}$. We plug in $x = a_n^{1/k}$ and $y = a^{1/k}$ to get

$$a_n - a = (a_n^{1/k} - a^{1/k}) \underbrace{\left(a_n^{\frac{k-1}{k}} + a_n^{\frac{k-2}{k}} a^{\frac{1}{k}} + \ldots + a_n^{\frac{1}{k}} a^{\frac{k-2}{k}} + a^{\frac{k-1}{k}}\right)}_{b_n}$$

We let b_n denote the second factor on the right hand side, so that we have

$$a_n - a = (a_n^{1/k} - a^{1/k})b_n.$$

By the triangle inequality, we have

$$b_n \geq a^{\frac{k-1}{k}} > 0.$$

Therefore we have

$$\frac{1}{b_n} \leq \frac{1}{a^{(k-1)/k}}$$

and so the sequence $\left\{\frac{1}{b_n}\right\}$ is bounded. We have

$$a_n^{1/k} - a^{1/k} = \frac{1}{b_n}(a_n - a)$$

and we know $a_n - a \to 0$, so by Lemma 1.3 we get that $a_n^{1/k} - a^{1/k} \to 0$.

It remains to consider the case when $a = 0$. We will do this directly. Fix $\varepsilon > 0$. As $a_n \to 0$, there exists a constant N such that if $n > N$ then $|a_n| < \varepsilon^k$. Thus, if $n > N$, then we have

$$|a_n^{1/k}| = (|a_n|)^{1/k} < \varepsilon.$$

This shows that $a_n^{1/k} \to 0$, as claimed. $\square$