Homework 9

Section 3.2

(9) Let c be a continuous function from $[0,1]$ to $[0,1]$. Prove that there is a point $c \in [0,1]$ such that $f(c) = c$, that is, show that f has a *fixed point*.

Solution: Following the hint in the book, we consider the function

$$g(x) = f(x) - x.$$

This is a function from $[0,1]$ to $\mathbf{R}$. We note that if $c \in [0,1]$, then $g(c) = 0$ if and only if $f(c) = c$. So, to prove f has a fixed point, we need to show that g has a zero on $[0,1]$. Consider the values of g at the endpoints 0 and 1. We have that $g(0) = f(0) - 0 = f(0)$. As $0 \leq f(0) \leq 1$, we have

$$0 \leq g(0) \leq 1.$$

If $g(0) = 0$, then $f(0) = 0$, and so we're done, as then 0 is a fixed point of f. So we might as well assume that $g(0) > 0$. Now consider $g(1) = f(1) - 1$. As $0 \leq f(1) \leq 1$, we get

$$-1 \leq g(1) \leq 0.$$

If $g(1) = 0$ then we are done, because this implies that $0 = g(1) = f(1) - 1$, so $f(1) = 1$, and therefore 1 is a fixed point of f. So we might as well assume that $g(1) < 0$. But now we have that $g(0) > 0$ is positive and $g(1) < 0$ is negative. By the Intermediate Value Theorem, there exists a point $c \in [0,1]$ such that $g(c) = 0$, and hence $f(c) = c$, so c is a fixed point of f.

(12) Use the Intermediate Value Theorem to prove that if f is a continuous function on an interval $[a, b]$ and if $f(x) \leq m$ for every $x \in [a, b)$, then $f(b) \leq m$.

Solution: We will prove the contrapositive. That is, we will show that if f is a continuous function on $[a, b]$ and $f(b) > m$, then there exists $x \in [a, b)$ such that $f(x) > m$. To prove this, consider an arbitrary point $y \in [a, b)$. If $f(y) > m$, we're done, so we might as well assume that $f(y) \leq m$. We're assuming that $f(b) > m$, so by the Intermediate Value Theorem, f assumes every value between $f(y)$ and $f(b)$ on the interval $[y, b]$. In particular, we have

$$m < \frac{f(b) + m}{2} < f(b)$$

and so there exists a point $x \in [y, b]$ such that $f(x) = \frac{f(b)+m}{2}$, and therefore

$$m < f(x) < f(b).$$

We can't have that $x = b$, because $f(x) < f(b)$, so we have $x \in [y, b)$, and in particular $x \in [a, b)$.

Section 4.1

(7) If $f(x) = \frac{\sin x}{|x|}$, find $\lim_{x \to 0^+} f(x)$ and $\lim_{x \to 0^-} f(x)$. Does $\lim_{x \to 0} f(x)$ exist?

Solution: We note that for $x > 0$, $f(x)$ agrees with the function $g(x) = \frac{\sin x}{x}$. Furthermore, the two–sided limit $\lim_{x \to 0} g(x)$ exists, and is equal to 1. Thus,

$$\lim_{x \to 0^+} \frac{\sin x}{|x|} = \lim_{x \to 0^+} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\sin x}{x} = 1.$$

For $x < 0$, $f(x)$ agrees with the function $-g(x) = \frac{-\sin x}{x}$. Thus, similarly to the above, we get that

$$\lim_{x \to 0^-} \frac{\sin x}{|x|} = \lim_{x \to 0^-} \frac{-\sin x}{x} = \lim_{x \to 0} \frac{-\sin x}{x} = -1.$$

Because the left and right limits do *not* agree, we conclude that $\lim_{x \to 0} \frac{\sin x}{|x|}$ does not exist.