Homework 8

Section 3.2

(2) Prove that if f is a continuous function on a closed bounded interval I and if $f(x)$ is never 0 for $x \in I$, then there is a number $m > 0$ such that $f(x) \geq m$ for all $x \in I$ or $f(x) \leq -m$ for all $x \in I$.

Solution: As $f(x) \neq 0$ for any $x \in I$, the Intermediate Value Theorem implies that either $f(x) > 0$ for every $x \in I$ or $f(x) < 0$ for every $x \in I$.
Suppose that $f(x) > 0$ for all $x \in I$. By Theorem 3.2.1, $f(x)$ attains its minimum on I, so there is some number say $a \in I$ such that $f(a) = m$ is the minimum value of $f(x)$ on I. Thus, $f(x) \geq m$ for all $x \in I$. Furthermore, as $a \in I$, our assumptions imply that $m = f(a) > 0$.
The case when $f(x) < 0$ for all $x \in I$ is similar. We apply Theorem 3.2.1 to get that $f(x)$ attains its maximum on I, so there is some number say $b \in I$ such that $f(b) = M$ is the maximum value of $f(x)$ on I. Thus, $f(x) \leq M$ for all $x \in I$. Furthermore, as $b \in I$, our assumptions imply that $M = f(b) < 0$. Thus, setting $m = -M$, we have that $m > 0$ and $f(x) \leq -m$ for all $x \in I$.

(5) Find an example of a function which is continuous on a closed (but not bounded) interval I but is not bounded. Then find an example of a function which is continuous and bounded on a closed interval I but does not have a maximum values.

Solution: For the first question, consider the interval $I = [0, \infty)$ and the function $f(x) = x$ on I. This is continuous on I, but not bounded.
For the second question, consider the interval $I = [1, \infty)$ and the function $f(x) = -\frac{1}{x}$ on I. This is continuous, and we have that if $x \geq 1$, then $|-1/x| = 1/x \leq 1$, so it is also bounded. But it doesn't have a maximum value: we have $f(x) < 0$ for any $x \in [1, \infty)$, and also if $\epsilon > 0$ then if $x > 1/\epsilon$ then we'll have that $f(x) = -1/x > -\epsilon$, so $-\epsilon$ is not an upper bound for $f(x)$.

(7) Give an example of a function defined on the interval $[0, 1]$ which does not take on every value between $f(0)$ and $f(1)$.

Solution: Consider the function

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq \frac{1}{2} \\ 1 & \text{if } \frac{1}{2} < x \leq 1. \end{cases}$$

The $f(0) = 0$ and $f(1) = 1$, but $f(x)$ only takes on the values 0 and 1 on the interval $[0, 1]$, and in particular does not take on the value (say) $\frac{1}{2}$.