Homework 7

Section 3.1

(4) Prove that $f(x) = |x|$ is continuous on all of $\mathbf{R}$.

Solution: Fix a real number a. We will show that $f(x)$ is continuous at a. Fix $\varepsilon > 0$. Let $\delta = \varepsilon$. Suppose that $x \in \mathbf{R}$ is a real number and

$$|x - a| < \delta = \varepsilon.$$

We apply the triangle inequality in the form

$$||u| - |v|| \leq |u - v|.$$

We get that

$$|f(x) - f(a)| = ||x| - |a|| \leq |x - a| \leq \varepsilon.$$

(6) Prove part **(d)** of Theorem 3.1.9. Explicitly, this is the following:

Theorem (Part **(d)** of Theorem 3.1.9)**.** Let f and g be real–valued function with domains of definition D_f and D_g. Let $a \in D_f \cap D_g$ be a real number such that f and g are both continuous at a and such that $g(a) \neq 0$. Then f/g is continuous at a.

Solution: We will use the sequential characterization of continuity (Theorem 3.1.6). Let $D \subset \mathbf{R}$ be the intersection of D_f, D_g, and the set of real numbers b such that $g(b) \neq 0$. Suppose that $\{x_n\}$ is a sequence of real numbers such that $x_n \in D$ for all n and such that $\lim x_n = a$. By Theorem 3.1.6, we have that

$$\lim f(x_n) = f(a) \quad \text{and} \quad \lim g(x_n) = g(a).$$

By the Main Limit Theorem, we have that

$$\lim \left(\frac{f(x_n)}{g(x_n)} \right) = \frac{f(a)}{g(a)}.$$

By Theorem 3.1.6, f/g is continuous at a.

(7) Prove Theorem 3.1.11. Explicitly, this is the following:

Theorem (Theorem 3.1.11)**.** Let f and g be real valued functions. Suppose that a is a real number which is in the domain of definition of $f \circ g$. Prove that if g is continuous at a and f is continuous at $g(a)$, then $f \circ g$ is continuous at a.

Solution: We will use the sequential characterization of continuity (Theorem 3.1.6). Suppose that $\{x_n\}$ is a sequence of real numbers such that x_n is in the domain of definition of $f \circ g$ for all n, and such that $\lim x_n = a$. As g is continuous at a, Theorem 3.1.6 implies that

$$\lim g(x_n) = g(a).$$

To make the notation clearer, lets set $y_n = g(x_n)$ and $b = g(a)$, so that the above equality becomes

$$\lim y_n = b.$$

As f is continuous at $b = g(a)$, Theorem 3.1.6 implies that

$$\lim f(y_n) = f(b).$$

Note that $f(y_n) = f(g(x_n)) = (f \circ g)(x_n)$ and $f(b) = f(g(a)) = (f \circ g)(a)$. Thus, we have shown that

$$\lim (f \circ g)(x_n) = (f \circ g)(a).$$

By Theorem 3.1.6, $f \circ g$ is continuous at a.

Bonus solution: Here is another solution to this problem, this time going directly from the definition of continuity. Fix $\varepsilon > 0$. Let's write down our assumptions. We first use that f is continuous at $g(a)$ to get that there exists a $\delta' > 0$ such that if y is in the domain of definition of f then

$$|y - g(a)| < \delta' \quad \text{implies} \quad |f(y) - f(g(a))| < \varepsilon. \tag{0.0.1}$$

We now use that g is continuous at a. Applying this with the epsilon being δ' (!), we get that there exists a $\delta > 0$ such that if x is in the domain of definition of g, then

$$|x - a| < \delta \quad \text{implies} \quad |g(x) - g(a)| < \delta'. \tag{0.0.2}$$

Now we put this together to prove the result. We claim that δ is the number we were looking for. Suppose that x is in the domain of definition of $f \circ g$, which means that x is in the domain of definition of g and $g(x)$ is in the domain of definition of f. If

$$|x - a| < \delta,$$

then by (0.0.2) we have that

$$|g(x) - g(a)| < \delta'.$$

Applying (0.0.1) with $y = g(x)$, we get that

$$|f(g(x)) - f(g(a))| < \varepsilon.$$

Thus, $f \circ g$ is continuous at a.