# Homework 6

## Section 2.4

**(9)** Prove that

$$\lim_{n\to\infty} \frac{2^n}{n} = \infty.$$

**Solution**: We will first prove that $n^2 \le 2^n$ for all $n \ge 4$ (note that this inequality is not true for $n = 3$!). We use induction on $n$. We check the base case $n = 4$ directly: this is the inequality $4^2 \le 2^4$, which is true. For the induction step, suppose that we already know that $n^2 \le 2^n$ for some $n \ge 4$. We then have that $(n-1)^2 \ge 2$, which can be rearranged to give

$$(n+1)^2 \le 2n^2.$$

Our induction hypothesis is that $n^2 \le 2^n$, which implies $2n^2 \le 2^{n+1}$. Combining these, we get

$$(n+1)^2 \le 2n^2 \le 2^{n+1}.$$

By induction, we have that $n^2 \le 2^n$ for all $n \ge 4$.
We now return to the problem at hand. We will show that $\lim_{n\to\infty} \frac{2^n}{n} = \infty$ directly from the definition. Let $M$ be a real number. Let $N$ be a real number such that $N > M$ and $N \ge 4$. If $n$ is a natural number such that $n > N$, the above bound $2^n \ge n^2$ implies

$$\frac{2^n}{n} \ge \frac{n^2}{n} = n > N > M.$$

This completes the proof.

**(11)** Prove part **(c)** of Theorem 2.4.7, which is the following:

**Theorem** (Part **(c)** of Theorem 2.4.7)**.** If $\{a_n\}$ is a sequence of real numbers, then $\lim_{n\to\infty} a_n = \infty$ if and only if $\lim_{n\to\infty}(-a_n) = -\infty$.

**Solution**: Suppose that $\lim_{n\to\infty} a_n = \infty$. We will show that $\lim_{n\to\infty}(-a_n) = -\infty$. Let $M$ be a real number. Applying the definition of the statement that $\lim_{n\to\infty} a_n = \infty$ to the constant $-M$, we get that there exists an $N$ such that for all $n > N$ we have $a_n > -M$. Thus, for all $n > N$ we have $-a_n < M$, so $\lim_{n\to\infty}(-a_n) = -\infty$.
Conversely, suppose that $\lim_{n\to\infty}(-a_n) = -\infty$. We will show that $\lim_{n\to\infty} a_n = -\infty$. Let $M$ be a real number. Applying the definition of the statement that $\lim_{n\to\infty}(-a_n) = -\infty$ to the constant $-M$, we get that there exists an $N$ such that for all $n > N$ we have $-a_n < -M$. Thus, for all $n > N$ we have $a_n > M$, so $\lim_{n\to\infty} a_n = \infty$.

**(12)** Prove part **(d)** of Theorem 2.4.7, which is the following:

**Theorem** (Part **(d)** of Theorem 2.4.7)**.** If $\{a_n\}$ and $\{b_n\}$ are sequences of real numbers and $a_n \leq b_n$ for all $n \in \mathbb{N}$, then $\lim_{n\to\infty} a_n = \infty$ implies $\lim_{n\to\infty} b_n = \infty$.

**Solution**: Assume that $\lim_{n\to\infty} a_n = \infty$. Let $M$ be a real number. Applying the definition of the statement that $\lim_{n\to\infty} a_n = \infty$, we get that there exists an $N$ such that for all $n > N$ we have $a_n > M$. It follows that $b_n \geq a_n > M$ for all $n > N$, so $\lim_{n\to\infty} b_n = \infty$.