Homework 5

Section 2.2

(5) Compute $\lim_{n\to\infty}\left(\sqrt{n^2+n}-n\right)$, and prove that your answer is correct.

Solution: We will show that
$$\lim_{n\to\infty}\left(\sqrt{n^2+n}-n\right)=\frac{1}{2}.$$
Before doing anything else, we record the following algebraic manipulation:
$$\sqrt{n^2+n}-n=\frac{\sqrt{n^2+n}+n}{\sqrt{n^2+n}+n}\left(\sqrt{n^2+n}-n\right)=\frac{(n^2+n)-n^2}{\sqrt{n^2+n}+n}=\frac{1}{1+\sqrt{1+\frac{1}{n}}}$$
Now we proceed with the proof. Fix $\epsilon>0$. Set
$$N=\frac{1}{1-(4\varepsilon-1)^2}.$$
Suppose that $n>N$. We then have
$$n>\frac{1}{1-(4\varepsilon-1)^2}$$
which rearranges to
$$1-\sqrt{1-\frac{1}{n}}<4\epsilon$$
(I found the above N by starting with this and solving for n. The reason why I wanted to start with the factor of 4 will become clear later). For future use, we also note that we have $1+\sqrt{1+\frac{1}{n}}>2$, and therefore
$$\frac{1}{1+\sqrt{1+\frac{1}{n}}}<\frac{1}{2}.$$
We therefore have
$$\left|\left(\sqrt{n^2+n}-n\right)-\frac{1}{2}\right|=\frac{1}{2}\cdot\frac{1-\sqrt{1-\frac{1}{n}}}{1+\sqrt{1+\frac{1}{n}}}<\frac{1}{2}\cdot\frac{4\epsilon}{2}=\epsilon.$$
This completes the proof.

(10) Give an example of a sequence $\{a_n\}$ which does not converge but for which the sequence $\{|a_n|\}$ does converge.

Solution: Consider the sequence defined by $a_n = (-1)^n$. We showed in lecture that $\{a_n\}$ does not converge. On the other hand, we have $|a_n| = 1$ for all n, so the sequence $\{|a_n|\}$ is simply the constant sequence 1, which converges to 1.

Section 2.3

(3) Use the main limit theorem to compute $\lim_{n\to\infty} \frac{2^n}{2^n+1}$.

Solution: Before doing anything else, we will show that $\lim_{n\to\infty} \frac{1}{2^n} = 0$ (we'll see why this is relevant soon). Indeed, say we fix $\epsilon > 0$. Set $N = \frac{1}{\epsilon}$. Because $n < 2^n$ for all $n \in \mathbb{N}$, we have that $\frac{1}{2^n} < \frac{1}{n}$ for all $n \in \mathbb{N}$. Now, if $n > N$, then we have $n > \frac{1}{\epsilon}$, and thus $\frac{1}{n} < \epsilon$. Putting this together, we get

$$\left|\frac{1}{2^n} - 0\right| = \frac{1}{2^n} < \frac{1}{n} < \epsilon.$$

Now let's go back to the problem at hand. We first note that

$$\frac{2^n}{2^n+1} = \frac{1}{1+\frac{1}{2^n}}.$$

(multiply the top and bottom of the right hand side by 2^n). We claim that we have

$$\lim_{n\to\infty} \frac{2^n}{2^n+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{2^n}} \overset{\text{(d)}}{=} \frac{\lim 1}{\lim\left(1+\frac{1}{2^n}\right)} \overset{\text{(b)}}{=} \frac{\lim 1}{\lim 1 + \lim \frac{1}{2^n}} = \frac{1}{1+0} = 1.$$

I'll justify each of the equals signs separately. The first equals sign is just the above algebraic manipulation, and the fifth is pretty obvious. The fourth equals sign is because of our above result that $\lim \frac{1}{2^n} = 0$, and the fact that $\lim 1 = 1$. The third equals sign is due to the main limit theorem part (b). Note that we have justified the assumptions of this theorem, because we have shown that the two sequences at hand (the constant sequence 1 and the sequence $\frac{1}{2^n}$ each converge). Similarly, the second equality is due to the main limit theorem part (d).