Homework 4

Section 2.1

(2) Use the triangle inequality to prove that there is no number x which satisfies both $|x-1| < 1/2$ and $|x-2| < 1/2$.

Solution: Suppose for the sake of contradiction that x is a real number such that $|x-1| < 1/2$ and $|x-2| < 1/2$. Here is the statement of the triangle inequality which we will use: if a and b are real numbers, then

$$|a+b| \leq |a| + |b|.$$

(see Theorem 2.1.2). We apply this with $a = x-1$ and $b = -x+2$. We get

$$1 = |(x-1)+(-x+2)| \leq |x-1| + |-x+2| = |x-1| + |x-2| < \frac{1}{2} + \frac{1}{2} = 1.$$

Thus we obtain $1 < 1$, which is absurd. We conclude that no such number x exists.

(5) Compute $\lim_{n\to\infty} \frac{2n-1}{3n+1}$, and prove that your answer is correct.

Solution: We will show that

$$\lim_{n\to\infty} \frac{2n-1}{3n+1} = \frac{2}{3}.$$

Fix $\epsilon > 0$. Set

$$N = \frac{1}{3}\left(\frac{5}{3\epsilon} - 1\right).$$

Say $n \in \mathbb{N}$ is a natural number such that $n > N$. We then have

$$n > \frac{1}{3}\left(\frac{5}{3\epsilon} - 1\right).$$

Rearranging by solving for ϵ, this gives

$$\frac{5}{3} \cdot \frac{1}{3n+1} < \epsilon.$$

We therefore have that

$$\left|\frac{2n-1}{3n+1} - \frac{2}{3}\right| = \frac{5}{3} \cdot \frac{1}{3n+1} < \epsilon$$

for all natural numbers $n > N$. Therefore the sequence converges to $2/3$, as claimed.

(11) Prove that if $a_n \to 0$ and k is any constant, then $ka_n \to 0$.

Solution: Let's first consider the case when $k = 0$. Then $ka_n = 0$ for all n, so $|a_n - 0| = 0$ for all n. Thus $a_n \to 0$.
Now we consider the case when $k \neq 0$. Choose $\epsilon > 0$. We apply the definition of "$a_n \to 0$" to the real number $\epsilon/|k|$ (note that because $\epsilon > 0$ and $|k| \geq 0$, we have $\epsilon/|k| > 0$ as well). The output is that there exists N such that for all $n > N$, we have

$$|a_n - 0| = |a_n| < \epsilon/|k|$$

Multiplying by $|k|$, we get that

$$|k| \cdot |a_n| = |ka_n| < \epsilon$$

for all $n > N$. Therefore $ka_n \to \infty$.