Homework 3

Section 1.4

(7) Prove that if $x < y$ are two real numbers, then there is a rational number r with $x < r < y$. Hint: Use the result of Example 1.4.9.

Solution: Following the hint, we will use the result of Example 1.4.9 (which we proved in lecture). This result is that if $a > 0$ is a positive real number, then there is a natural number $n \in \mathbb{N}$ such that $1/n < a$. We assume $x < y$, so $0 < y - x$. We apply Example 1.4.9 to get that there is a natural number $n \in \mathbb{N}$ such that

$$0 < 1/n < y - x.$$

Rearranging this, we get

$$nx < nx + 1 < ny.$$

Set $m = \lfloor nx \rfloor$. Here, the right hand side is the largest integer which is $\leq nx$ (this is called the *floor function*). We have

$$nx - 1 < m \leq nx,$$

and therefore

$$nx < m + 1 \leq nx + 1.$$

Combining this with our previous inequalities, we get

$$nx < m + 1 \leq nx + 1 < ny.$$

We now divide by n (and forget about the $nx + 1$ term) to get

$$x < \frac{m+1}{n} < y.$$

The rational number $(m + 1)/n$ therefore has the properties we were looking for.

(8) Prove that if x is irrational and r is a non-zero rational number, then $x + r$ and rx are also irrational.

Solution: Say $r = \frac{m}{n}$ for two integers m, n with $n \neq 0$.
Suppose for the sake of contradiction that $x + r$ were rational. Then we would be able to write

$$x + r = \frac{a}{b}$$

for some integers a, b such that $b \neq 0$. Subtracting r from both sides, this becomes

$$x = \frac{a}{b} - r = \frac{a}{b} - \frac{m}{n} = \frac{an - bm}{bn}.$$

But this implies that x is rational, which contradicts our assumption that x was irrational. We conclude that $x + r$ is irrational.
Suppose for the sake of contradiction that rx were rational. Then we would be able to write

$$rx = \frac{a}{b}$$

for some integers a, b such that $b \neq 0$. We assumed that r was nonzero, so we can divide by it, yielding

$$x = \frac{a}{b}\frac{n}{m} = \frac{an}{bm}.$$

This shows x is rational, which again contradicts our assumption that it was not. We conclude that rx is irrational.

(9) We know that $\sqrt{2}$ is irrational. Use this fact and the previous exercise to prove that if $r < s$ are rational numbers, then there is an irrational number x with $r < x < s$.

Solution: We apply Example 1.4.9 to find a natural number $n \in \mathbb{N}$ such that

$$0 < \frac{1}{n} < s - r$$

(note that $s - r$ is indeed positive). Rearranging, this becomes

$$r < r + \frac{1}{n} < s.$$

We have that $0 < \sqrt{2} < 2$, which implies that $0 < \frac{\sqrt{2}}{2} < 1$, so

$$0 < \frac{\sqrt{2}}{2n} < \frac{1}{n}.$$

Combining this with our previous inequality, we get

$$r < r + \frac{\sqrt{2}}{2n} < r + \frac{1}{n} < s.$$

Using that $\sqrt{2}$ is irrational and problem (8), we get that the numbers $\frac{\sqrt{2}}{2n}$ and $r + \frac{\sqrt{2}}{2n}$ are irrational. Thus, $r + \frac{\sqrt{2}}{2n}$ has the properties we were looking for.