Homework 2

Section 1.4

(1) For each of the following sets, describe the set of all upper bounds for the set:

(a) of all odd integers;

(b) $\{1 - 1/n | n \in \mathbb{N}\}$;

(c) $\left\{r \in \mathbb{Q} | r^3 < 8\right\}$;

(d) $\{\sin x | x \in \mathbb{R}\}$.

Solution: (a) : For every real number x, there exists an odd integer n such that $n > x$. Thus, the given set has no upper bound, and so the set of all upper bounds is the empty set.

(b) : If $n \in \mathbb{N}$, then we have $1 - 1/n < 1$. Furthermore, if x is a real number such that $x < 1$, then there exists $n \in \mathbb{N}$ such that $x < 1 - 1/n$. Thus, the set of all upper bounds for the set is

$$\{x \in \mathbb{R} | x \geq 1\}.$$

(c) : We first describe the given set more concretely. Say $r \in \mathbb{Q}$. If $r \leq 0$, then $r^3 < 8$ is always true. If $r \geq 0$, then $r^3 < 8$ if and only if $r < \sqrt[3]{8}$. Thus, the given set is equal to

$$\left\{r \in \mathbb{Q} | r < \sqrt[3]{8}\right\}.$$

We can now see that the set of all upper bounds for this set is equal to

$$\left\{x \in \mathbb{R} | x \geq \sqrt[3]{8}\right\}.$$

(d) : For any real number x, we have that $-1 \leq \sin x \leq 1$. Furthermore, the extreme values are achieved, for instance by $x = \pi/2$ and $x = 3\pi/2$. Thus, the given set is equal to the interval $[-1, 1]$. Hence, the set of all upper bounds is

$$\{x \in \mathbb{R} | x \geq 1\}.$$

(4) Show that the set

$$A = \left\{x | x^2 < 1 - x\right\}$$

is bounded above, and then find its least upper bound.

Solution: Suppose that $x^2 < 1 - x$. This implies that $x^2 + x - 1 < 0$. We factor the left hand side to get the inequality

$$x^2 + x - 1 = \left(x - \frac{-1+\sqrt{5}}{2}\right)\left(x - \frac{-1-\sqrt{5}}{2}\right) < 0.$$

The only way a product of two real numbers can be negative is if exactly one of the numbers is negative. The first term is negative if and only if

$$x < \frac{-1+\sqrt{5}}{2}$$

and the second term is negative if and only if

$$x < \frac{-1-\sqrt{5}}{2}.$$

We also note that

$$\frac{-1-\sqrt{5}}{2} < \frac{-1+\sqrt{5}}{2},$$

so if the second term is negative then the first is automatically also negative. We conclude that the inequality $x^2 < 1 - x$ holds if and only if

$$\frac{-1-\sqrt{5}}{2} < x < \frac{-1+\sqrt{5}}{2}.$$

In particular, this shows that A is bounded above, and that the least upper bound for A is the real number $(-1+\sqrt{5})/2$.

(10) Show that if L_x and L_y are Dedekind cuts defining real numbers x and y, then

$$L_x + L_y = \{r + s | r \in L_x \text{ and } s \in L_y\}$$

is also a Dedekind cut (this is the Dedekind cut defining $x + y$).

Solution: We are trying to show that $L_x + L_y$ is a Dedekind cut, so we need to verify that the three conditions of Definition 1.4.1 hold for $L_x + L_y$.
(1) : Because L_x and L_y are Dedekind cuts, we know that $L_x \neq \emptyset$ and $L_y \neq \emptyset$. Thus, there exist elements $r \in L_x$ and $s \in L_y$. Then $r + s \in L_x + L_y$, showing that $L_x + L_y \neq \emptyset$. We next show that $L_x + L_y \neq \mathbb{Q}$. As L_x is a Dedekind cut, we know that $L_x \neq \mathbb{Q}$. Thus, we may find a rational number $a \in \mathbb{Q}$ such that $a \notin L_x$. We may similarly find a rational number b such that $b \notin L_y$. We will show that $a + b \notin L_x + L_y$. Indeed, if $r \in L_x$ and $s \in L_y$, then by condition (3) we have that $a > r$ and $b > s$. Thus $a + b > r + s$ for any $r \in L_x$ and $s \in L_y$. It follows that $a + b \notin L_x + L_y$.

(2) : Consider elements $r \in L_x$ and $s \in L_y$. We will show that the element $r + s \in L_x + L_y$ is not a largest element of $L_x + L_y$. Indeed, as L_x and L_y are Dedekind cuts, by condition (2) we can find $a \in L_x$ such that $r < a$ and $b \in L_y$ such that $s < b$. Then $r + s < a + b$, and $a + b \in L_x + L_y$. Thus $r + s$ is not the largest element of $L_x + L_y$. As r and s were arbitrary, this shows that $L_x + L_y$ has no largest element.

(3) : Consider elements $r \in L_x$ and $s \in L_y$. Let $t \in \mathbb{Q}$ be a rational number such that $t < r + s$. We need to show that $t \in L_x + L_y$. The inequality $t < r + s$ implies that $t - s < r$, so by property (3) we have that $t - s \in L_x$. We also have $s \in L_y$. Thus, the equality

$$t = (t - s) + s \in L_x + L_y$$

shows that $t \in L_x + L_y$.