Homework 1

Section 1.2

(8) Using induction, prove that $7^n - 2^n$ is divisible by 5 for every $n \in \mathbb{N}$.

Solution: Base case ($n = 1$): we have that $7^1 - 2^1 = 7 - 2 = 5$, which is divisible by 5.
Induction step: we assume that $7^n - 2^n$ is divisible by 5. We have that

$$7^{n+1} - 2^{n+1} = (5 + 2) \cdot 7^n - 2 \cdot 2^n = 5 \cdot 7^n + 2 \cdot (7^n - 2^n).$$

Both of these terms are divisible by 5, so $7^{n+1} - 2^{n+1}$ is divisible by 5. This completes the induction step.

(9) Using induction, prove that $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$.

Solution: Base case ($n = 1$): we compute that

$$\sum_{k=1}^{1} k = 1 = \frac{1(1+1)}{2}.$$

Induction step: suppose we already know that

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

for some particular value of n. We compute that

$$\sum_{k=1}^{n+1} k = \sum_{k=1}^{n} k + (n+1) = \frac{n(n+1)}{2} + n + 1 = \frac{n^2 + n + 2n + 2}{2} = \frac{n^2 + 3n + 2}{2} = \frac{(n+1)(n+2)}{2}.$$

Thus the desired equality holds also for $n + 1$. This completes the induction step.

(13) Let a sequence $\{x_n\}$ of numbers be defined recursively by

$$x_1 = 1 \quad \text{and} \quad x_{n+1} = \frac{1}{1 + x_n}.$$

Prove by induction that x_{n+2} is between x_n and x_{n+1} for each $n \in \mathbb{N}$.

Section 1.3

The two problems in this section use the following definition.

Definition 1.3.1: A *commutative ring* is a set R with two binary operations, *addition* $(a, b) \mapsto a + b$ and *multiplication* $(a, b) \mapsto ab$, that satisfy the following axioms:

A1. (Commutative Law of Addition) $x + y = y + x$ for all $x, y \in R$.

A2. (Associative Law of Addition) $x + (y + z) = (x + y) + z$ for all $x, y, z \in R$.

A3. (Additive Identity) There is an element $0 \in R$ such that $0 + x = x$ for all $x \in R$.

A4. (Additive Inverses) For each $x \in R$, there is an element $-x \in R$ such that $x + (-x) = 0$.

M1. (Commutative Law of Multiplication) $xy = yx$ for all $x, y \in R$.

M2. (Associative Law of Multiplication) $x(yz) = (xy)z$ for all $x, y, z \in R$.

M3. (Multiplicative Identity) There is an element $1 \in R$ such that $1 \neq 0$ and $1x = x$ for all $x \in R$.

D. (Distributive Law) $x(y + z) = xy + xz$ for all $x, y, z \in R$.

(3) Prove that if $\mathbb{Z}$ satisfies the axioms for a commutative ring, then $\mathbb{Q}$ satisfies **A1** and **M1**.

Solution: We check **A1**. Consider elements $a/b, c/d \in \mathbb{Q}$, where $a, b, c, d \in \mathbb{Z}$ and $b, d \neq 0$. Using the definition of addition in $\mathbb{Q}$, we compute

$$\frac{a}{b} + \frac{c}{d} \overset{\text{def}}{=} \frac{ad + bc}{bd} \overset{\textbf{(M1)}}{=} \frac{da + cb}{db} \overset{\textbf{(A1)}}{=} \frac{cb + da}{db} \overset{\text{def}}{=} \frac{c}{d} + \frac{a}{b}.$$

Here, in the middle two equals signs we have used respectively that **(M1)** and **(A1)** hold in $\mathbb{Z}$. The first and last equals signs are the definition of addition in $\mathbb{Q}$.
We now check **M1**. Consider elements $a/b, c/d \in \mathbb{Q}$, where $a, b, c, d \in \mathbb{Z}$ and $b, d \neq 0$. Using the definition of multiplication in $\mathbb{Q}$, we compute

$$\frac{a}{b} \cdot \frac{c}{d} \overset{\text{def}}{=} \frac{ac}{bd} \overset{\textbf{(M1)}}{=} \frac{ca}{db} \overset{\text{def}}{=} \frac{c}{d} \cdot \frac{a}{b}.$$

(4) Prove that if $\mathbb{Z}$ satisfies the axioms for a commutative ring, then $\mathbb{Q}$ satisfies **A2** and **M2**.