Homework 12

1 Chapter 5.2

(9) Prove that if f is integrable on $[a,b]$, then so is f^2.

Solution: As f is integrable, it is (by definition) bounded. Let M be a number such that $|f(x)| \leq M$ for all $x \in [a,b]$. Following the hint, we will show that

$$|f^2(x) - f^2(y)| \leq 2M|f(x) - f(y)| \tag{1.1}$$

for all $x, y \in [a,b]$. We first note that the inequality $|f(x)| \leq M$ implies that

$$|f(x) + f(y)| \leq |f(x)| + |f(y)| \leq M + M = 2M.$$

Here we used the triangle inequality. Now, we have

$$|f^2(x) - f^2(y)| = |(f(x) + f(y))(f(x) - f(y))| = |f(x) + f(y)| \cdot |f(x) - f(y)| \leq 2M|f(x) - f(y)|.$$

This proves the claim.

We now prove the result. Let $P = \{x_0, \ldots, x_n\}$ be a partition of $[a,b]$. Still following the hint, we will attempt to bound $U(f^2, P) - L(f^2, P)$ in terms of $U(f, P) - L(f, P)$. As usual, we set $I_k = [x_{k-1}, x_k]$ for $k = 1, \ldots, n$. We consider the above inequality (1.1). We take the supremum of both sides over all $x, y \in I_k$. This gives us the inequality

$$\sup_{I_k} f^2 - \inf_{I_k} f^2 \leq 2M(\sup_{I_k} f - \inf_{I_k} f). \tag{1.2}$$

We now consider the lower and upper sums for f and f^2 with respect to the partition P. Note: because we are considering the functions f and f^2 at the same time, we *won't* use the usual notation m_k and M_k in the lower and upper sums. Using (1.2), we get the following chain of inequalities.

$$\begin{aligned}
U(f^2, P) - L(f^2, P) &= \sum_{k=1}^{n} (\sup_{I_k} f^2)(x_k - x_{k-1}) - \sum_{k=1}^{n} (\inf_{I_k} f^2)(x_k - x_{k-1}) \\
&= \sum_{k=1}^{n} (\sup_{I_k} f^2 - \inf_{I_k} f^2)(x_k - x_{k-1}) \\
&\leq \sum_{k=1}^{n} 2M(\sup_{I_k} f - \inf_{I_k} f)(x_k - x_{k-1}) \\
&= 2M \sum_{k=1}^{n} (\sup_{I_k} f - \inf_{I_k} f)(x_k - x_{k-1}) \\
&= 2M(U(f, P) - L(f, P)).
\end{aligned}$$

By Theorem 5.1.8, because f is integrable, we can find a sequence $\{P_n\}$ of partitions of $[a, b]$ such that

$$\lim_{n\to\infty} U(f, P_n) - L(f, P_n) = 0.$$

We just showed above that for *any* partition P we have

$$U(f^2, P) - L(f^2, P) \le 2M(U(f, P) - L(f, P)).$$

It follows that

$$\lim_{n\to\infty} U(f^2, P_n) - L(f^2, P_n) = 2M \lim_{n\to\infty} U(f, P) - L(f, P) = 0.$$

By Theorem 5.1.8, f^2 is integrable on $[a, b]$.

(10) Prove that if f and g are integrable on $[a, b]$, then so is fg.

Solution: By Theorem 5.2.3 part **(b)**, the function $f + g$ is integrable on $[a, b]$. By **(9)**, the functions f^2, g^2, and $(f + g)^2$ are all integrable on $[a, b]$. By Theorem 5.2.3 part **(a)**, the functions $-f^2$ and $-g^2$ are also integrable on $[a, b]$, so by Theorem 5.2.3 part **(b)** the function

$$(f + g)^2 + (-f^2) + (-g^2) = 2fg$$

is integrable on $[a, b]$. To finish, we apply Theorem 5.2.3 part **(a)** to get that $\frac{1}{2} \cdot (2fg) = fg$ is integrable on $[a, b]$.

(11) Give an example of a function f such that $|f|$ is integrable on $[0, 1]$, but f is not integrable on $[0, 1]$.

Solution: Consider the function f on $\mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ -1 & \text{if } x \text{ is irrational.} \end{cases}$$

We have that $|f(x)| = 1$ for any x, so $|f|$ just the constant function 1. In particular, $|f|$ is integrable on $[0, 1]$. Let's show however that f itself is *not* integrable on $[0, 1]$. This is going to be very similar to an example we did in lecture. We note that, for any interval $[a, b]$ with $a < b$, there is both a rational number x and an irrational number y in between a and b. Thus, we have that the supremum of f on $[a, b]$ is equal to 1, and the infimum of f on $[a, b]$ is equal to 0. Now say that $P = \{x_0, \ldots, x_n\}$ is a partition of $[0, 1]$. We compute that

$$U(f, P) = \sum_{k=1}^{n} M_k(x_k - x_{k-1}) = \sum_{k=1} 1 \cdot (x_k - x_{k-1}) = 1$$

while

$$L(f, P) = \sum_{k=1}^{n} m_k(x_k - x_{k-1}) = \sum_{k=1} (-1) \cdot (x_k - x_{k-1}) = -1.$$

It follows that the upper integral of f over $[0, 1]$ is equal to 1, while the lower integral of f over $[0, 1]$ is equal to -1. These are not the same, so f is not integrable on $[0, 1]$.