Homework 11

Section 5.1

(1) Find the upper sum $U(f,P)$ and lower sum $L(f,P)$ if $f(x) = 1/x$ on $[1,2]$ and P is the partition of $[1,2]$ into four subintervals of equal length.

Solution: The partition P is $\left\{1, \frac{5}{4}, \frac{3}{2}, \frac{7}{4}, 2\right\}$. The function $1/x$ is monotonically decreasing on $[1,2]$, so the supremum of $1/x$ on each interval is the left endpoint, and the infimum is the right endpoint. Thus, the upper sum is

$$U(f,P) = \frac{1}{1} \cdot \frac{1}{4} + \frac{1}{5/4} \cdot \frac{1}{4} + \frac{1}{3/2} \cdot \frac{1}{4} + \frac{1}{7/4} \cdot \frac{1}{4} = \frac{319}{420}.$$

The lower sum is

$$L(f,P) = \frac{1}{5/4} \cdot \frac{1}{4} + \frac{1}{3/2} \cdot \frac{1}{4} + \frac{1}{7/4} \cdot \frac{1}{4} + \frac{1}{2} \cdot \frac{1}{4} = \frac{533}{840}.$$

(2) Prove that $\int_0^1 x\,\mathrm{d}x$ exists by computing $U(f, P_n)$ and $L(f, P_n)$ for the function $f(x) = x$ and a partition P_n of $[0, 1]$ into n equal subintervals. Then show that condition (5.1.7) of Theorem 5.1.8 is satisfied. Calculate the integral by taking the limit of the upper sums. Hint: Use Exercise 1.2.9.

Solution: We consider the partition

$$P_n = \left\{0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}, 1\right\}.$$

The lower and upper sums are

$$U(f, P_n) = \sum_{k=1}^{n} \frac{k}{n} \cdot \frac{1}{n} = \frac{1}{n^2} \sum_{k=1}^{n} k$$

and

$$L(f, P_n) = \sum_{k=1}^{n} \frac{k-1}{n} \cdot \frac{1}{n} = \frac{1}{n^2} \sum_{k=1}^{n} (k-1).$$

Using the fact that $\sum_{k=1}^{n} k = n(n+1)/2$, we get

$$U(f, P_n) = \frac{1}{n^2} \cdot \frac{n(n+1)}{2} = \frac{1}{2}\left(1 + \frac{1}{n}\right)$$

and

$$L(f, P_n) = \frac{1}{n^2} \cdot \frac{(n-1)n}{2} = \frac{1}{2}\left(1 - \frac{1}{n}\right).$$

We compute that

$$U(f, P_n) - L(f, P_n) = \frac{1}{2}\left(1 + \frac{1}{n}\right) - \frac{1}{2}\left(1 - \frac{1}{n}\right) = \frac{1}{n}.$$

Thus we have

$$\lim_{n\to\infty} \left(U(f, P_n) - L(f, P_n)\right) = \lim_{n\to\infty} \frac{1}{n} = 0$$

so condition (5.1.7) of Theorem 5.1.8 holds. Therefore

$$\int_0^1 x\,\mathrm{d}x = \lim_{n\to\infty} U(f, P_n) = \lim_{n\to\infty} \frac{1}{2}\left(1 + \frac{1}{n}\right) = \frac{1}{2}.$$

(5) Let f be the function on $[0, 1]$ which is 0 at every rational number and 1 at every irrational number. Is this function integrable on $[0, 1]$? Prove that your answer is correct by using the definition of the integral.

Solution: Suppose $P = \{x_0, \ldots, x_n\}$ is a partition of $[0, 1]$. We know that on every interval $[x_{k-1}, x_k]$ there is some irrational number a such that $x_{k-1} < a < x_k$ and some rational number b such that $x_{k-1} < b < x_k$. Thus, f takes the value 0 and the value 1 on each interval $[x_{k-1}, x_k]$. Therefore, $M_k = 1$ and $m_k = 0$ for every k. It follows that $U(f, P) = 1$ and $L(f, P) = 0$. This holds for every partition P, so we have

$$\overline{\int_0^1} f(x)\,\mathrm{d}x = 1 \qquad \text{and} \qquad \underline{\int_0^1} f(x)\,\mathrm{d}x = 0.$$

These are different numbers, so therefore f is *not* integrable on $[0, 1]$.