Homework 10

Section 4.2

(1) Using just the definition of the derivative, show that the derivative of $1/x$ is $-1/x^2$.

Solution: We compute

$$\lim_{x\to a}\frac{1/x-1/a}{x-a}=\lim_{x\to a}\frac{a-x}{ax(x-a)}=\lim_{x\to a}\frac{\text{-1}}{xa}=\frac{\text{-1}}{a^2}.$$

Thus, $f'(a)=-1/a^2$ for all a, so $f'(x)=-1/x^2$.

(3) Show how to derive the expression for the derivative of $\tan x$ if you know the derivatives of $\sin x$ and $\cos x$.

Solution: Using the quotient rule, we compute

$$\tan' x=\left(\frac{\sin x}{\cos x}\right)'=\frac{\cos x\sin' x-\cos' x\sin x}{\cos^2 x}.$$

Thus, if we know $\sin' x$ and $\cos' x$, this gives us an expression for $\tan' x$.

(12) Is the function defined by

$$f(x) = \begin{cases} x^2 & \text{if } x > 0, \\ 0 & \text{if } x \leq 0 \end{cases}$$

differentiable at 0?

Solution: The answer is yes. Indeed, by definition, f being differentiable at a means that the limit

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

exists. We compute this limit for $a = 0$ by first computing the two one–sided limits. This looks like:

$$\lim_{x \to 0^+} \frac{f(x) - f(0)}{x} = \lim_{x \to 0^+} \frac{f(x)}{x} = \lim_{x \to 0^+} \frac{x^2}{x} = \lim_{x \to 0^+} x = 0$$

and

$$\lim_{x \to 0^-} \frac{f(x) - f(0)}{x} = \lim_{x \to 0^-} \frac{0 - 0}{x} = 0.$$

As both one–sided limits exists and are equal (!), the two–sided limit also exists (and is equal to the same number). Thus, f is differentiable at 0. We have also shown that $f'(0) = 0$, by the way.