1 Answers to sample problems

1.

$$
\begin{aligned}
|S \cup T \cup U| &= \{1, 2, 3, 4, 5, 6, 7\} \\
|S \cap T| &= \{3, 5\} \\
|S \cap U| &= \{2, 3\} \\
|T \cap U| &= \{3\} \\
|S \cap T \cap U| &= \{3\}
\end{aligned}
$$

The inclusion exclusion equation is

$$7 = 5 + 2 + 4 - 2 - 2 - 1 + 1$$

It works!

2. Let S be the set of integers between 1 and 10000 which are divisible by 2 and let T be the set which are divisible by 5. We have

$$S = \{2, 4, \dots, 10000\} \qquad T = \{5, 10, \dots, 10000\}$$

so $|S| = 5000$ and $|T| = 2000$. We have that $S \cap T$ is the set of integers which are divisible by 10, so

$$S \cap T = \{10, 20, \dots, 10000\}$$

and $|S \cap T| = 1000$. The inclusion exclusion equation is

$$|S \cup T| = |S| + |T| - |S \cap T| = 5000 + 2000 - 1000 = 6000.$$

Therefore the answer is $10000 - 6000 = 4000$.

3. Let S be the set of integers between 1 and 10000 which are divisible by 2, T the set which are divisible by 3, and U the set which are divisible by 5. Then $S \cap T$ is the set which are divisible by 6, $S \cap U$ is the set which are divisible by 10, $T \cap U$ is the set which are divisible by 15, and $S \cap T \cap U$ is the set which are divisible by 30. We have

$$
\begin{aligned}
|S| &= 5000 \\
|T| &= 3333 \\
|U| &= 2000 \\
|S \cap T| &= 1666 \\
|S \cap U| &= 1000 \\
|T \cap U| &= 666 \\
|S \cap T \cap U| &= 333
\end{aligned}
$$

We get

$$|S \cup T \cup U| = 5000 + 3333 + 2000 - 1666 - 1000 - 666 + 333 = 7334$$

This is our answer.

4. We need to show that $\sim$ is reflexive, symmetric, and transitive.

- Reflexive: suppose that $(a, b) \in \mathbb{R} \times \mathbb{R} - \{(0, 0)\}$. We have $(a, b) = (1 \cdot a, 1 \cdot b)$, so $(a, b) \sim (a, b)$.

- Symmetric: suppose that $(a,b),(c,d) \in \mathbb{R}\times\mathbb{R}-\{(0,0)\}$ and that $(a,b)\sim(c,d)$. This means that $(a,b)=(\lambda c,\lambda d)$ for some nonzero $\lambda\in\mathbb{R}$. We notice that this implies $(c,d)=(\lambda^{-1}a,\lambda^{-1}b)$, and therefore $(c,d)\sim(a,b)$.
- Transitive: suppose that $(a,b),(c,d),(e,f)\in\mathbb{R}\times\mathbb{R}-\{(0,0)\}$ and that $(a,b)\sim(c,d)$ and $(c,d)\sim(e,f)$. Then we have

  $$(a,b)=(\lambda c,\lambda d) \qquad \text{and} \qquad (c,d)=(\mu e,\mu f)$$

  for some nonzero numbers $\lambda,\mu\in\mathbb{R}$. It follows that

  $$(a,b)=(\lambda\mu e,\lambda\mu f),$$

  and therefore $(a,b)\sim(e,f)$.

5. Equivalence relations are in bijection with partitions. There are 203 partitions of a set with 6 elements. Relations are the same as subsets of the Cartesian product. The product $\{1,2,3,4,5,6\}\times\{1,2,3,4,5,6\}$ has 36 elements, so there are 2^{36} possible relations.
6. Suppose that $g\circ f$ is 1-1. Suppose that $s,s'\in S$ and $f(s)=f(s')$. We take g of both sides to get $g(f(s))=g(f(s'))$, or equivalently $(g\circ f)(s)=(g\circ f)(s')$. As $g\circ f$ is 1-1, this implies that $s=s'$. Therefore, f is 1-1. Suppose that $g\circ f$ is onto. Take $u\in U$. There exists $s\in S$ such that $(g\circ f)(s)=u$. Then we have $g(f(s))=u$, so g sends the elements $f(s)\in T$ to $u\in U$, and therefore g is onto.
7. Suppose that f is 1-1. Say $S=\{s_1,\dots,s_n\}$. Then the n elements $f(s_1),\dots,f(s_n)\in S$ are all distinct. But S only has n elements, so these must be all the elements of S, and therefore f is onto. Suppose that f is onto. Then we have $\{f(s_1),\dots,f(s_n)\}=S$. As S has n elements, the elements $f(s_1),\dots,f(s_n)$ must all be distinct, to f is 1-1.
8. The function $f:\mathbb{Z}\to\mathbb{Z}$ defined by $f(x)=2x$ is 1-1, but not onto. The function $g:\mathbb{Z}\to\mathbb{Z}$ defined by $g(x)=\lfloor x/2\rfloor$ is onto, but not 1-1.
9. We have $f(f^{-1}(x))=x$ for all $x\in\mathbb{R}$. Plugging in $f(x)=7x-3$, we get

   $$x=f(f^{-1}(x))=7f^{-1}(x)-3.$$

   Solve for $f^{-1}(x)$ to get

   $$f^{-1}(x)=\frac{1}{7}(x+3).$$

10. If n is even, then $1^n=(-1)^n$, so $f(1)=f(-1)$. Therefore $f(x)$ is not 1-1, and hence not bijective.

    For the other half of the problem, we will use the following key fact: if n is even, then there are two solutions to $x^n=1$ (namely $x=\pm1$), and if x is odd, then there is only one solution to $x^n=1$ (namely $x=1$). Now, suppose that n is odd. We will show that f is bijective. To see that it is 1-1, suppose that $x,y\in\mathbb{R}$ and $f(x)=f(y)$. This means that $x^n=y^n$. If $y=0$, then we must also have $x=0$, so suppose that $y\neq0$. Then this equation can be rearranged to give $(x/y)^n=1$. By the key fact, we have $x/y=1$, so $x=y$. Therefore f is 1-1. Next let's show f is onto. Choose $y\in\mathbb{R}$. We want to find an $x\in\mathbb{R}$ such that $f(x)=y$, or equivalently $x^n=y$. If $y\geq0$, the existence of such an x is given by Proposition 4.1 in the book (this proposition says that there exists an nth root of any real number which is ≥0). Suppose that $y\leq0$. Then $-y\geq0$, so we can apply the proposition to get an $x\in\mathbb{R}$ such that $x^n=-y$. Now we use that n is odd, and therefore $(-x)^n=-x^n=y$. So, $f(-x)=y$, and therefore f is onto. We have shown that f is 1-1 and onto, so f is therefore bijective.

11. We have to solve

$$f(x)^2 + 2f(x) + 1 = e^{2x}$$

Rearrange this to get

$$(f(x) + 1)^2 = e^{2x}$$

This implies

$$f(x) + 1 = \pm e^x$$

so

$$f(x) = e^x - 1, \qquad f(x) = -e^x - 1$$

are the two possibilities.

12. The cycle notation for f is

$$f = (1\,4)(2\,6\,3\,8\,9)(5)(7\,10)$$

Remember it doesn't matter what order you write the cycles in, so for instance

$$f = (2\,6\,3\,8\,9)(1\,4)(7\,10)$$

also works. The cycle type of f is

$$(5, 2, 2, 1) = (5, 2^2, 1)$$

and the order of f is the least common multiple of $(5, 2, 2, 1)$, which is 10. To get the inverse of f, we take the mirror image of each of the cycles, which gives

$$f^{-1} = (9\,8\,3\,6\,2)(4\,1)(10\,7)$$

As before, there are lots of ways to write this. For instance, we also have

$$f^{-1} = (2\,9\,8\,3\,6)(1\,4)(7\,10)$$

13. I worked these out by hand. Here is my list:

$$\begin{aligned}
&(1, 1, 1, 1, 1, 1)\\
&(2, 1, 1, 1, 1)\\
&(2, 2, 1, 1)\\
&(2, 2, 2)\\
&(3, 1, 1, 1)\\
&(3, 2, 1)\\
&(3, 3)\\
&(4, 1, 1)\\
&(4, 2)\\
&(5, 1)\\
&(6)
\end{aligned}$$

There are 11 possible cycle types.

14. We know that the order of a permutation is the least common multiple of the lengths of the cycles. We worked out the possible cycle types for an element of S_6 in the previous problem. Looking at the cases, the largest order we can get is 6. This happens for cycle type $(3, 2, 1)$, because $\mathrm{lcm}(3, 2, 1) = 6$, and for cycle type (6), because $\mathrm{lcm}(6) = 6$. For example, here are two permutations with order 6:

$$(1\,2\,3)(4\,5)(6) \qquad (1\,2\,3\,4\,5\,6)$$

15. An element of S_5 of order 3 must have cycle type $(3, 1, 1)$. Such a permutation is determined by choosing the three numbers to go in the 3-cycle, and then choosing one of the $3! = 6$ possible orderings of the three numbers in the 3-cycle. But, the resulting permutation only depends on the ordering of the elements of the 3-cycle up to cyclic permutation, so there are only $3!/3 = 6/3 = 2$ possible permutations for a given choice of elements in the 3–cycle. For instance, if we choose $1, 2, 3$ to go in the 3–cycle, then the two permutations will be

$$(1\,2\,3)(4)(5) \quad \text{and} \quad (1\,3\,2)(4)(5)$$

and any other choice of ordering for the numbers 1,2,3 gives the same permutation as one of these. We conclude that there are $\binom{5}{3} \cdot 2 = 20$ permutations in S_5 with cycle type $(3, 1, 1)$, hence 20 permutations with order 3.

16. For the first one, we can take any 3-cycle and its inverse. For instance, if

$$f = (1\,2\,3)(4)(5)(6) \quad \text{and} \quad g = (1\,3\,2)(4)(5)(6),$$

then we have $fg = (1)(2)(3)(4)(5)(6)$, so fg has order 1. For the others, I found examples by playing around with examples. Here's what I found: if

$$f = (1\,2\,3)(4)(5)(6) \quad \text{and} \quad g = (2\,3\,4)(1)(5)(6),$$

then

$$fg = (1\,2)(3\,4)(5)(6)$$

so fg has order 2. If

$$f = (1\,2\,3)(4)(5)(6) \quad \text{and} \quad g = (4\,5\,6)(1)(2)(3),$$

then

$$fg = (1\,2\,3)(4\,5\,6)$$

so fg has order 3. If

$$f = (1\,2\,3)(4)(5)(6) \quad \text{and} \quad g = (3\,4\,5)(1)(2)(6),$$

then

$$fg = (1\,2\,3\,4\,5)(6)$$

so fg has order 5.