Review for Final Exam (Math 2200, Spring 2023)

The final exam is cumulative. This review sheet only covers the content which is not on the review sheets for the first or second midterms.

1 Sample Problems

1. Let $S = \{1, 2, 3, 4, 5\}$, $T = \{3, 5\}$, and $U = \{2, 3, 6, 7\}$. Find $|S \cup T \cup U|$, $|S \cap T|$, $|S \cap U|$, $|T \cap U|$, and $|S \cap T \cap U|$, and verify that the inclusion exclusion principle equation

$$|S \cup T \cup U| = |S| + |T| + |U| - |S \cap T| - |S \cap U| - |T \cap U| + |S \cap T \cap U|$$

holds.

2. Find the number of integers between 1 and 10000 which are divisible by neither 2 nor 5.

3. Find the number of integers between 1 and 10000 which are divisible by at least one of $2, 3$, or 5.

4. Write out a detailed proof that the following relation on the set $\mathbb{R} \times \mathbb{R} - \{(0, 0)\}$ is an equivalence relation:

$$(a, b) \sim (c, d) \iff \text{there exists a nonzero } \lambda \in \mathbb{R} \text{ such that } (a, b) = (\lambda c, \lambda d).$$

Describe the equivalence classes geometrically.

5. How many equivalence relations are there on the set $S = \{1, 2, 3, 4, 5, 6\}$? How many relations are there on S?

6. Let $f : S \to T$ and $g : T \to U$ be functions. Prove that if $g \circ f$ is 1-1 then f is 1-1. Prove that if $g \circ f$ is onto then g is onto.

7. Let S be a finite set and let $f : S \to S$ be a function. Show that if f is 1-1 then f is bijective. Also show that if f is onto then f is bijective.

8. Show that the finiteness in the previous problem is really needed. That is, find an example of a set S and a function $f : S \to S$ which is 1-1 but is not onto, and also find an example which is onto but is not 1-1.

9. Show that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 7x - 3$ is bijective. Find a formula for f^{-1}.

10. Let $n \geq 1$ be a positive integer. Show that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^n$ is bijective if and only if n is odd.

11. Say $f, g : \mathbb{R} \to \mathbb{R}$ are two functions, and we know that

$$g(x) = x^2 + 2x + 1 \qquad \text{and} \qquad (g \circ f)(x) = e^{2x}.$$

Find the possibilities for $f(x)$.

12. Write the following permutation in cycle notation:

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 4 & 6 & 8 & 1 & 5 & 3 & 10 & 9 & 2 & 7 \end{pmatrix}$$

What is the cycle type of f? What is the order of f? Write f^{-1} in cycle notation.

13. How many possible cycle types are there for an element of S_6?
14. What is the largest possible order of an element of S_6?
15. How many permutations are there in S_5 which have order 3?
16. Find examples of two permutations $f, g \in S_6$ such that f and g each have order 3 and fg has order d for each of $d = 1, 2, 3, 5$.

2 Review topics by chapter

Chapter §16

- Know the definition of the multinomial coefficients.
- Know the formula for the multinomial coefficients (Proposition 16.5)

Chapter §17

- Know the definition of the union and the intersection of two sets.
- Know the definition of the Cartesian product of two sets.
- Know the statement of the inclusion exclusion principle for two and three sets (Proposition 17.2 and formula 17.1).

Chapter §18

- Know the definition of an equivalence relation.
- Know the definition of an equivalence class.
- Know the key fact that the equivalence classes of an equivalence relation on a set S form a partition of S (Proposition 18.1)

Chapter §19

- Know the definition of a function.
- Know the definition of the *domain*, *codomain*, and *image* of a function.
- Know what it means for a function to be *one-to-one*, *surjective*, and *bijective*.
- Know the definition of the composition of two functions.
- Know the definition of the inverse of a function.
- Know the key fact that a function has an inverse if and only if it is a bijection.

Chapter §20

- Know the definition of a permutation.

- Know how to write a permutation in cycle notation.
- Know what the cycle type of a permutation is. Be able to list all possible cycle types for elements of S_n for $n = 1, 2, 3, 4, 5, 6$.
- Given two permutations in cycle notation, know how to compute their composition. Also know how to compute the inverse of a permutation given in cycle notation.
- Know what the order of a permutation is, and how to compute it when given the permutation in cycle notation (Proposition 20.4).
- Be able to write out a list of all elements of S_3 or S_4, along with their cycle types and orders.