Midterm 2 Math 2200 Spring 2023

Directions:

- You may cite results proved during lecture or in the book without repeating the proof.
- If you wish to use a result from lecture or from the book, you must write out the complete statement which you are citing.

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
Total:	50	

1. (10 points) For each part, circle either true or false. You do not have to justify your answer. In each part, m is a positive integer and x, y, z, n, k are integers.

(A)	If $xy \equiv 0 \pmod{m}$, then either $x \equiv 0 \pmod{m}$ or $y \equiv 0 \pmod{m}$.	**TRUE**	**<u>FALSE</u>**
(B)	If $x + z \equiv y + z \pmod{m}$, then $x \equiv y \pmod{m}$.	**<u>TRUE</u>**	**FALSE**
(C)	If $m \nmid z$ and $xz \equiv yz \pmod{m}$, then $x \equiv y \pmod{m}$.	**TRUE**	**<u>FALSE</u>**
(D)	If x is not divisible by m, then $x^{m-1} \equiv 1 \pmod{m}$.	**TRUE**	**<u>FALSE</u>**
(E)	If $0 \leq k \leq n$ then $\binom{n}{k-1} \leq \binom{n}{k}$	**TRUE**	**<u>FALSE</u>**

2. (10 points) Let p be a prime number. If x is an integer such that $x^2 \equiv 1 \pmod{p}$, show that $x \equiv 1 \pmod{p}$ or $x \equiv -1 \pmod{p}$.

Solution: Suppose that $x^2 \equiv 1 \pmod{p}$. Subtracting, this means that $x^2 - 1 \equiv 0 \pmod{p}$, or equivalently that $(x-1)(x+1) \equiv 0 \pmod{p}$. This is equivalent to the statement that p divides $(x-1)(x+1)$. As p is prime, this implies that either p divides $x-1$ or p divides $x+1$. In the first case, we have $x - 1 \equiv 0 \pmod{p}$, so $x \equiv 1 \pmod{p}$, and in the second case we have $x + 1 \equiv 0 \pmod{p}$, so $x \equiv -1 \pmod{p}$.

Comment: You really need the assumption that p is prime for this result to be true. For instance, say $m = 15$. Then $x = 1, 4, 11, 14$ all satisfy $x^2 \equiv 1 \pmod{15}$. (note that $14 \equiv -1 \pmod{15}$, but 4 and 11 are not congruent to either 1 or -1 modulo 15).

3. (10 points) Find a solution to the congruence equation $7x^5 \equiv 3 \pmod{23}$. You do **not** have to simplify your answer (ie. you can leave it as a sum/product of integers). You may find it helpful to use the equalities

$$10 \cdot 7 - 3 \cdot 23 = 1 \qquad \text{and} \qquad 9 \cdot 5 - 2 \cdot 22 = 1.$$

Solution: The first equation above implies that $10 \cdot 7 \equiv 1 \pmod{23}$. Thus, multiplying both sides of the equation $7x^5 \equiv 3 \pmod{23}$ by 10 yields the equivalent equation $10 \cdot 7x^5 \equiv 10 \cdot 3 \pmod{23}$, which simplifies to

$$x^5 \equiv 7 \pmod{23}.$$

To solve this for x, we use the method from class. The second equation above implies that $9 \cdot 5 \equiv 1 \pmod{22}$, and therefore a solution to the equation is 7^9. This is a fine answer to the question. If we want a smaller solution, we can take this modulo 23, which turns out to be $x = 15$.

4. (10 points) Find the coefficient of x in the expansion of $(2\sqrt{x} + \frac{3}{x^2})^{12}$.

Solution: By the binomial theorem, the terms of the expansion look like

$$\binom{12}{a}(2\sqrt{x})^{12-a}(\frac{3}{x^2})^a$$

where a ranges over the integers $0, 1, 2, \ldots, 12$. The power of x in the above term is $x^{\frac{12-a}{2}-2a} = x^{\frac{12-5a}{2}}$. So, to get $x = x^1$, we need

$$\frac{12-5a}{2} = 1$$

Solving this we get $a = 2$. In particular, only one of the above terms in the expansion contributes. Plugging in $a = 2$, we get that the x term of the expansion is

$$\binom{12}{2}(2\sqrt{x})^{12-2}(\frac{3}{x^2})^2 = \binom{12}{2}2^{10}3^2x.$$

So the answer to the question is

$$\binom{12}{2}2^{10}3^2 = 66 \cdot 1024 \cdot 9 = 608256.$$

(any of these forms is fine!)

5. In this question we will consider words with letters coming from the set $\{a, b, c, d, e\}$. In each part, you can give your answer as a sum/product of integers and binomial coefficients/factorials (you do not need to simplify at all).

   (a) (3 points) How many words are there with exactly eight letters? For example, here are three of them:

   $$bbcbeeeb, abecdabc, bbcbbeee$$

   Solution: There are 5 choices for each of the eight letters in a word, so by the multiplication principle there are $5^8 = 390625$ words.

   (b) (3 points) How many words are there with exactly eight letters which use the letter b exactly twice?

   Solution: There are $\binom{8}{2}$ possible choices of locations for the two b's. After making this choice, each of the remaining six letters can be any of a, c, d, e. So, the total number of possible words is

   $$\binom{8}{2} \cdot 4^6 = 28 \cdot 4096 = 114688$$

   (c) (4 points) How many words are there with exactly eight letters which use each of the letters a, b, c exactly once?

   Solution: There are $\binom{8}{3}$ choices of three locations to put a, b, c, and for each of these choices, there are $3! = 6$ ways of placing a, b, c. For the remaining five letters in the word, we can put in the remaining letters d, e however we like. So, the answer is

   $$\binom{8}{3} \cdot 6 \cdot 2^5 = 56 \cdot 6 \cdot 32 = 10752.$$