Midterm 1 Math 2200 Spring 2023

Directions:

- You may cite results proved during lecture or in the book without repeating the proof.
- If you wish to use a result from lecture or from the book, you must write out the complete statement which you are citing.

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
Total:	50	

1. (10 points) For each part, circle either true or false. You do not have to justify your answer. In each part a, b, and c are integers.

(A)	$\{1,2\} \in \{1, \{2,3\}, 2\}$	**TRUE**	**<u>FALSE</u>**
(B)	$\{2,3\} \in \{1, \{2,3\}, 2\}$	**<u>TRUE</u>**	**FALSE**
(C)	$\{1,2\} \subseteq \{1, \{2,3\}, 2\}$	**<u>TRUE</u>**	**FALSE**
(D)	If $ab \mid c$ then $a \mid c$.	**<u>TRUE</u>**	**FALSE**
(E)	If $a \mid bc$ then $a \mid b$ and $a \mid c$.	**TRUE**	**<u>FALSE</u>**

2. Give **examples** of each of the following. You do not have to justify your answers at all.

   (a) (3 points) An infinite set of irrational numbers.

   Solution: $\{n + \sqrt{2} | n \in \mathbb{Z}\}$

   (b) (3 points) Irrational numbers $x, y \in \mathbb{R}$ such that $x + y$ and xy are both **rational**.

   Solution: $x = \sqrt{2}, y = -\sqrt{2}$

   (c) (4 points) Integers a, b, and c such that $a|bc$ but $a \nmid b$ and $a \nmid c$. (Here, "$x \nmid y$" means "x does not divide y").

   Solution: $a = 6, b = 2, c = 3$

3. (10 points) Using induction, prove that for every $n \geq 1$ we have

$$1 \cdot 2^1 + 2 \cdot 2^2 + 3 \cdot 2^3 + 4 \cdot 2^4 + \cdots + n \cdot 2^n = (n-1)2^{n+1} + 2.$$

(Expressed in terms of Σ–notation, the left hand side is $\sum_{k=1}^{n} k2^k$)

Solution:

Base case ($n = 1$):

$$1 \cdot 2^1 = (1-1)2^{1+1} + 2$$

Induction step: Assume that the formula holds for some $n \geq 1$. In other words, assume that

$$1 \cdot 2^1 + 2 \cdot 2^2 + 3 \cdot 2^3 + 4 \cdot 2^4 + \cdots + n \cdot 2^n = (n-1)2^{n+1} + 2.$$

Add $(n+1) \cdot 2^{n+1}$ to both sides to get

$$1 \cdot 2^1 + 2 \cdot 2^2 + \cdots + n \cdot 2^n + (n+1) \cdot 2^{n+1} = (n-1)2^{n+1} + 2 + (n+1)2^{n+1}.$$

We compute

$$\begin{aligned}(n-1)2^{n+1} + 2 + (n+1)2^{n+1} &= 2n \cdot 2^{n+1} + 2 \\ &= n2^{n+2} + 2\end{aligned}$$

Therefore

$$1 \cdot 2^1 + 2 \cdot 2^2 + \cdots + n \cdot 2^n + (n+1) \cdot 2^{n+1} = n2^{n+2} + 2$$

so the formula holds for $n+1$. By induction, the formula holds for all $n \geq 1$.

4. (10 points) Prove that $\sqrt[3]{2}$ is irrational.

Solution 1: Suppose, for the sake of contradiction, that $\sqrt[3]{2} = \frac{a}{b}$ for two integers a, b with $b \neq 0$. We may assume that both a and b are positive. Rearranging this equation and cubing, this implies

$$2b^3 = a^3.$$

By the fundamental theorem of arithmetic, there are integers a' and b' which are not divisible by 2 such that $a = 2^m a'$ and $b = 2^n b'$ for some $n, m \geq 0$. The above equation implies

$$2^{3n+1} b'^3 = 2^{3m} a'^3.$$

Both a'^3 and b'^3 are not divisible by 2. Therefore the prime factorization of the left hand side contains 2 to the power $3n+1$ and the prime factorization of the right hand side contains 2 to the power $3m$. The fundamental theorem of arithmetic says that these powers have to be equal. But it can never be the case that $3n + 1 = 3m$ for two integers m and n. This is a contradiction. We conclude that $\sqrt[3]{2}$ is irrational.

Solution 2: Here is another way that a lot of you did this. Suppose, for the sake of contradiction, that $\sqrt[3]{2} = \frac{a}{b}$ for two integers a, b with $b \neq 0$. We may assume that both a and b are positive. We may further assume that a and b are not both even, because if they were we could keep cancelling 2's. Rearranging this equation and cubing, we get

$$2b^3 = a^3.$$

This implies that a^3 is even. As 2 is prime, $2|a^3$ implies $2|a$, so a is also even. This means that $a = 2k$ for some integer k. Substituting this in, we get

$$2b^3 = (2k)^3 = 8k^3$$

and therefore

$$b^3 = 4k^3$$

This means that $4|b^3$. Therefore b^3 is even, so as before b is even. But this is a contradiction, because we assumed that a and b were not both even. We conclude that $\sqrt[3]{2}$ is irrational.

5. (10 points) Let a, b, and n be integers. Prove that

$$\mathrm{hcf}(na, nb) = n \cdot \mathrm{hcf}(a, b)$$

Solution 1: Set

$$d = \mathrm{hcf}(a, b) \qquad \text{and} \qquad e = \mathrm{hcf}(na, nb)$$

We want to show that $nd = e$.
On the one hand, as d is a common factor of a and b, we have $d|a$ and $d|b$, so $nd|na$ and $nd|nb$. It follows that $nd|\mathrm{hcf}(na, nb) = e$.
On the other hand, let $e = \mathrm{hcf}(na, nb)$. By a result proved in lecture, every common factor of na and nb has to divide e, so in particular n divides e. Therefore we can write $e = nf$ for some integer f. We know that $nf|na$ and $nf|nb$, so $f|a$ and $f|b$. Therefore $f|\mathrm{hcf}(a, b) = d$ and so $e = nf|nd$.
We have shown that $nd|e$ and $e|nd$. As they are both positive, we conclude that $nd = e$.

Solution 2: The idea here is to use the fundamental theorem of arithmetic. Say

$$\begin{aligned} a &= p_1^{\alpha_1} \cdots p_m^{\alpha_m} \\ b &= p_1^{\beta_1} \cdots p_m^{\beta_m} \\ n &= p_1^{\gamma_1} \cdots p_m^{\gamma_m} \end{aligned}$$

where $p_1 < \cdots < p_m$ are prime numbers and $\alpha_i \geq 0$, $\beta_i \geq 0$, and $\gamma_i \geq 0$. By a result proved in lecture, we have

$$\mathrm{hcf}(a, b) = p_1^{\min(\alpha_1, \beta_1)} \cdots p_m^{\min(\alpha_m, \beta_m)}$$

We have

$$\begin{aligned} na &= p_1^{\alpha_1+\gamma_1} \cdots p_m^{\alpha_m+\gamma_m} \\ nb &= p_1^{\beta_1+\gamma_1} \cdots p_m^{\beta_m+\gamma_m} \end{aligned}$$

so by the same result we have

$$\mathrm{hcf}(na, nb) = p_1^{\min(\alpha_1+\gamma_1, \beta_1+\gamma_1)} \cdots p_m^{\min(\alpha_m+\gamma_m, \beta_m+\gamma_m)}$$

We have that $\min(\alpha_i + \gamma_i, \beta_i + \gamma_i) = \min(\alpha_i, \beta_i) + \gamma_i$, so

$$\begin{aligned} \mathrm{hcf}(na, nb) &= p_1^{\min(\alpha_1, \beta_1)+\gamma_1} \cdots p_m^{\min(\alpha_m, \beta_m)+\gamma_m} \\ &= (p_1^{\gamma_1} \cdots p_m^{\gamma_m}) \left(p_1^{\min(\alpha_1, \beta_1)} \cdots p_m^{\min(\alpha_m, \beta_m)} \right) \\ &= n \, \mathrm{hcf}(a, b) \end{aligned}$$

Solution 3: The idea here is to apply the Euclidean algorithm to find the highest common factors of (a, b) and (na, nb), and compare the steps. We can assume that $a < b$. First, let's say we apply the Euclidean algorithm to (a, b), getting

$$\begin{aligned} b &= q_1 a + r_1 \\ a &= q_2 r_1 + r_2 \\ r_1 &= q_3 r_2 + r_3 \\ &\vdots \\ r_{m-3} &= q_{m-1} r_{m-2} + r_{m-1} \\ r_{m-2} &= q_m r_{m-1} + r_m \\ r_{m-1} &= q_{m+1} r_m + 0 \end{aligned}$$

Then $r_m = \text{hcf}(a, b)$. Now let's apply the Euclidean algorithm to (na, nb), getting

$$\begin{aligned} nb &= q_1'(na) + r_1' \\ na &= q_2' r_1' + r_2' \\ r_1' &= q_3' r_2' + r_3' \\ &\vdots \\ r_{s-3}' &= q_{s-1}' r_{s-2}' + r_{s-1}' \\ r_{s-2}' &= q_s' r_{s-1}' + r_s' \\ r_{s-1}' &= q_{s+1}' r_s' + 0 \end{aligned}$$

We get $r_s' = \text{hcf}(na, nb)$. Here, I've used r' and q' because we don't know yet that these have anything to do with the previous r and q. I've also used s as the step at which we stop, because we don't know yet that s is equal to m.
Now let's compare the two. Dividing the first equation in the second list by n and comparing with the first equation in the first list, we get

$$q_1 a + r_1 = b = q_1' a + \frac{r_1'}{n}$$

This shows that $n | r_1'$, so r_1'/n is an integer. Furthermore, we have $0 \leq r_1' < na$, so $0 \leq r_1'/n < a$. By the uniqueness of the remainder, we get $r_1 = r_1'/n$. Continuing in this way, we get $r_2 = r_2'/n, \ldots, r_m = r_m'/n$, and $r_{m+1} = r_{m+1}'/n$. As $r_{m+1} = 0$, we get that $m = s$, and the final nonzero remainders are r_m and r_m'. We know $r_m = r_m'/n$, so $\text{hcf}(a, b) = \text{hcf}(na, nb)/n$, and therefore $n \cdot \text{hcf}(a, b) = \text{hcf}(na, nb)$.