HW 9

March 30, 2023

Problem 0.1 (Chapter 16, problem 5)**.**

(a) How many words of ten or fewer letters can be formed using the alphabet $\{a, b\}$?

(b) Using the alphabet $\{a, b, c, d, e, f\}$, how many six letter words are there that use all six letters, in which no two of the letters a, b, c occur consecutively?

Proof. Part (a): The number of words with exactly n letters is equal to 2^n. So the answer is

$$2^1 + 2^2 + 2^3 + 2^4 + 2^5 + 2^6 + 2^7 + 2^8 + 2^9 + 2^{10}$$

Another way of expressing this is to use the formula for the finite geometric series:

$$\frac{x^{n+1} - 1}{x - 1} = 1 + x + x^2 + \cdots + x^n$$

Taking $x = 2$ and doing some rearranging, this gives the above number to be

$$2^{11} - 2$$

Part (b): None of a, b, c can go next to each other, so the word we are building must look like

$$\_ * \_ * \_ * \_$$

where $* = d, e, f$. We only have three letters to put in the blanks, so the fourth blank will not get a letter. There are 4! ways to put in a, b, c into the blanks, and 3! ways to substitute d, e, f for the $*$'s. So, the answer is $4! \cdot 3! = 24 \cdot 6 = 144$.

$\square$

Problem 0.2 (Chapter 16, problem 6)**.**

(a) Find the number of arrangements of the set $\{1, 2, \ldots, n\}$ in which the numbers $1, 2$ appear as neighbors.

(b) Let $n \geq 5$. Find the number of arrangements of the set $\{1, 2, \ldots, n\}$ in which the numbers $1, 2, 3$ appear as neighbors in order, and so do the numbers $4, 5$.

Proof. Part (a): Let's count these arrangements as follows. First, pick the block of two slots where 1 and 2 occur. There are $n - 1$ different choices for this. Then, pick the order in which 1 and 2 appear. There are two possible orders. Finally, pick the arrangement for the remaining $n - 2$ numbers. There are $(n - 2)!$ possible arrangements. So, the answer is

$$(n - 1) \cdot 2 \cdot (n - 2)! = 2(n - 1)!$$

Part (b): To count these, note that $1, 2, 3$ appear as a block, and so do $4, 5$. Thus, the answer is the number of arrangements of $n - 3$ objects, which is $(n - 3)!$. $\square$

Problem 0.3 (Chapter 16, problem 8b)**.** Prove that for any positive integer n,

$$3^n = \sum_{k=0}^{n} \binom{n}{k} 2^k$$

Proof. Here is a counting proof. We will show that both sides are equal to the numbers of ways of choosing subsets $S \subset T \subset \{1, 2, \ldots, n\}$. On the one hand, we could count these pairs by the size of T. For each $k = 0, \ldots, n$, there are $\binom{n}{k}$ different subsets $T \subset \{1, 2, \ldots, n\}$ of size k, and given a subset T of size k, there are 2^k different subsets $S \subset T$. So, the number of such pairs of subsets where T has size k is $\binom{n}{k}2^k$. Thus, the total number of pairs is

$$\sum_{k=0}^{n} \binom{n}{k} 2^k$$

On the other hand, to specify such a pair of subsets, we can say for each element $1, 2, \ldots, n$ if that element is either in both S and T, just in T, or in neither. So, the number of such pairs of subsets is 3^n. Putting this together, we conclude that

$$3^n = \sum_{k=0}^{n} \binom{n}{k} 2^k$$

$\square$