HW 3

March 30, 2023

Problem 0.1 (Chapter 8, problem 2)**.** Prove by induction that for any integer $n \geq 1$ we have

$$\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$$

Proof. Base case: $r = 1$.

$$1^2 = \frac{1}{6}1 \cdot (1+1) \cdot (2+1)$$

Induction step: assume for some n that

$$\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$$

Add $(n+1)^3$ to both sides

$$\sum_{r=1}^{n} r^2 + (n+1)^2 = \frac{1}{6}n(n+1)(2n+1) + (n+1)^2$$

The LHS is

$$\sum_{r=1}^{n+1} r^2$$

We rearrange the RHS to get

$$\frac{1}{6}n(n+1)(2n+1) + (n+1)^2 = \frac{1}{6}(n+1)(n+2)(2n+3)$$

This shows the identity holds also for $n+1$. By induction, it holds for all $n \geq 1$. $\square$

Problem 0.2 (Chapter 8, problem 5a)**.** Prove by induction that for all integers $n \geq 0$ the number $5^{2n} - 3^n$ is a multiple of 11.

Proof. Base case: $n = 0$. We have

$$5^0 - 3^0 = 0$$

This is equal to 0 times 11.

Induction step: assume for some n that $5^{2n} - 3^n$ is divisible by 11. This means that

$$5^{2n} - 3^n = 11k$$

for some integer k. Now lets look at the number $5^{2(n+1)} - 3^{n+1}$. We compute

$$\begin{aligned} 5^{2(n+1)} - 3^{n+1} &= 25 \cdot 5^{2n} + 3 \cdot 3^n \\ &= (2 \cdot 11 + 3)5^{2n} + 3 \cdot 3^n \\ &= 11 \cdot (2 \cdot 5^{2n}) + 3(5^{2n} + 3^n) \\ &= 11 \cdot (2 \cdot 5^{2n}) + 3 \cdot 11k \\ &= 11 \cdot (2 \cdot 5^{2n} + 3k) \end{aligned}$$

Here, in the second to last step we used the induction hypothesis. We have shown that $5^{2(n+1)} - 3^{n+1}$ is divisible by 11. By induction, $5^{2n} - 3^n$ is divisible by 11 for all $n \geq 0$. $\square$

Problem 0.3 (Chapter 8, problem 5e)**.** Prove by induction that if $n \geq 3$ is an integer then

$$5^n > 4^n + 3^n + 2^n$$

Proof. Base case: $n = 3$. We compute

$$5^3 > 4^3 + 3^3 + 2^3$$

The LHS is 125, and the RHS is $64 + 27 + 8 = 99$.

Induction step: suppose for some $n \geq 3$ that we have

$$5^n > 4^n + 3^n + 2^n$$

We have

$$\begin{aligned} 5^{n+1} &= 5 \cdot 5^n \\ &> 5 \cdot (4^n + 3^n + 2^n) \\ &= 5 \cdot 4^n + 5 \cdot 3^n + 5 \cdot 2^n \\ &> 4 \cdot 4^n + 3 \cdot 3^n + 2 \cdot 2^n \\ &= 4^{n+1} + 3^{n+1} + 2^{n+1} \end{aligned}$$

Therefore the inequality holds also for $n + 1$. By induction, it holds for all $n \geq 3$. $\square$