HW 13

Problem 0.1 (Chapter 20, problem 3)**.**

(a) List the numbers that occur as the orders of elements of S_4, and calculate how many elements there are in S_4 of each of these orders.

(b) List all possible cycle–shapes of even permutations in S_6.

(c) Calculate the largest possible order of any permutation in S_{10}.

(d) Calculate the largest possible order of any even permutation in S_{10}.

(e) Find a value of n such that S_n has an element of order greater than n^2.

Proof.
(a) : During lecture I worked out a table of possible cycle types of elements of S_4, together with the numbers of elements of each type and the orders. Here it is:

cycle type	example	how many	order
$(1,1,1,1)$	$(1)(2)(3)(4)$	1	1
$(2,1,1)$	$(1\,2)(3)(4)$	6	2
$(2,2)$	$(1\,2)(3\,4)$	3	2
$(3,1)$	$(1\,2\,3)(4)$	8	3
(4)	$(1\,2\,3\,4)$	6	4

Adding up the ones with a given order we get that there is 1 element with order 1, 9 with order 2, 8 with order 3, and 6 with order 4.

(b) : By Proposition 20.7, a permutation is even if and only if it has an even number of cycles of even length. Working through the possibilities, I get the following list:

$$(1,1,1,1,1,1)$$
$$(2,2,1,1)$$
$$(3,1,1,1)$$
$$(3,3)$$
$$(4,2)$$
$$(5,1)$$

(c) : The order of a permutation is the least common multiple of the lengths of the cycles. So, we need to find the maximum value of $\operatorname{lcm}(a_1,\dots,a_n)$ where $a_1,\dots,a_n$ are positive integers such that $a_1+\cdots+a_n=10$. I claim that the maximum we can get is the partition $5+3+2=10$, for which we have $\operatorname{lcm}(5,3,2)=30$. For instance, the permutation

$$(1\,2\,3\,4\,5)(6\,7\,8)(9\,10)$$

has order 30. To see that this really is the biggest we can get, we could write down every possible cycle type for an element of S_{10}, work out the order for each cycle type, and then check if 30 is the biggest. It turns out there are 42 different possible cycle types for elements of S_{10} (ie. there are 42 partitions of 10). This isn't too crazy big to work out by hand. We can cut down our work a bit though by noticing that if $f \in S_{10}$ has all cycles of length ≤ 4, then the order of f is the least common multiple of a bunch of integers which are ≤ 4. But we have $\text{lcm}(1, 2, 3, 4) = 12$, so the order of f is ≤ 12, and therefore we can ignore these cycle types. For the rest, I'll just write them all down. Here are the possibilities when f has a cycle of length 5:

cycle type	order
(5,1,1,1,1,1)	5
(5,2,1,1,1)	10
(5,2,2,1)	10
(5,3,1,1)	15
(5,3,2)	30
(5,4,1)	20
(5,5)	5

Here are the possibilities if f has a cycle of length 6:

cycle type	order
(6,1,1,1,1)	6
(6,2,1,1)	6
(6,3,1)	6
(6,2,2)	6
(6,4)	12

Finally, here are all the possibilities if f has a cycle of length $7, 8, 9$, or 10.

cycle type	order
(7,1,1,1)	7
(7,2,1)	14
(7,3)	21
(8,1,1)	8
(8,2)	8
(9,1)	9
(10)	10

We conclude that 30 is the largest possible order of an element of S_{10}. Note that we have also shown that any such element has to have cycle type $(5, 3, 2)$.

(d) : I'll use my tables above of all cycle types with a cycle of length ≥ 5. As before, a permutation is even if and only if it has an even number of cycles of even length. Including only these cycle types, we get the following list:

cycle type	order
(5,1,1,1,1,1)	5
(5,2,2,1)	10
(5,3,1,1)	15
(5,5)	5
(6,2,1,1)	6
(6,4)	12
(7,1,1,1)	7
(7,3)	21
(8,2)	8
(9,1)	9

The biggest order which shows up in the above table is 21, for cycle type $(7,3)$. For example,

$$(1\,2\,3\,4\,5\,6\,7)(8\,9\,10)$$

is an even permutation in S_{10} and has order 21. Note that, as before, any cycle type with all cycles of length ≤ 4 has order ≤ 12 so 21 is the maximum order of an even permutation in S_{10}.

(e) : Here is my idea for this: to get an element in S_n of really big order, we should consider a cycle type which looks like $(p_1, \ldots, p_m)$ for distinct prime numbers p_i. For example, in part (c) above, we looked at $n = 10$ and the cycle type $(2,3,5)$, which had order $2 \cdot 3 \cdot 5 = 30$. In general, the order of an element f of cycle type $(p_1, \ldots, p_m)$ where $p_1, \ldots, p_m$ are distinct primes is $p_1 \cdot \ldots \cdot p_m$. On the other hand, we have $f \in S_n$ where $n = p_1 + \cdots + p_m$. The product grows a lot faster than the sum, so by taking enough primes we should get something that works. So, I tried taking $p_1, \ldots, p_m$ to be the first m prime numbers, and computed the following table.

cycle type	element of	order
(2)	S_2	2
(2,3)	S_5	6
(2,3,5)	S_{10}	30
(2,3,5,7)	S_{17}	210
(2,3,5,7,11)	S_{28}	2310

We have that $28^2 = 784 < 2310$. So, any permutation in S_{28} with cycle type $(2,3,5,7,11)$ will have order 2310, and hence have order $> 28^2$. Thus, $n = 28$ is one possible answer to the question. $\square$

Problem 0.2 (Chapter 20, problem 7ab)**.** Let S be a set of size m and T a set of size n. Assume that $m \geq n$.

(a) What is the number of onto functions from S to T if $m = n$?

(b) Show that if $m = n + 1$, the number of onto functions from S to T is

$$\binom{n+1}{2} \cdot n!$$

Proof.
(a) : If $m = n$, then any onto function $f : S \to T$ is automatically 1-1. To see this, say $S = \{s_1, \ldots, s_m\}$. As f is onto, we have $T = \{f(s_1), \ldots, f(s_m)\}$. But T has exactly m elements, so the elements $f(s_1), \ldots, f(s_m) \in T$ must all be distinct. Thus f is 1-1. This shows that if $m = n$ then the number of onto functions from S to T is the same as the number of bijective functions from S to T. We can count these using the multiplication principle. To define a function $f : S \to T$, we need to

pick $f(s_1), \dots, f(s_m)$. There are m possibilities for $f(s_1)$. We want f to be 1-1, so we can't send s_2 to $f(s_1)$, and there are therefore $m-1$ possible choices for $f(s_2)$. Similarly, there are $m-2$ choices for $f(s_3)$, and so on. We conclude that there are

$$m \cdot (m-1) \cdot (m-2) \cdot \ldots \cdot 1 = m!$$

onto functions from S to T.

(b) : For a function $f : S \to T$ to be onto, there must be exactly two elements of S which get sent to the same element of T, and all the other elements of S must be sent to distinct elements of T. Thus, f must look like this:

$$\begin{array}{ccccccccccc}
S & = & \{a, & \dots & b, & x, & & y, & c, & \dots & d\} \\
{\scriptstyle f}\big\downarrow & & \big\downarrow & & \big\downarrow & & \searrow \; \swarrow & & \big\downarrow & & \big\downarrow \\
T & = & \{t_1, & \dots & t_{k-1}, & & t_k, & & t_{k+1}, & \dots & t_n\}
\end{array}$$

(in this picture, the elements of S are listed in some arbitrary order. I'm not saying that the first element of S has to go to the first element of T, or anything like that). So, we can determine every onto function $f : S \to T$ as follows: first choose two elements of S, say $x, y \in S$, and one element of T, say $t_k \in T$, and declare that $f(x) = t_k$ and $f(y) = t_k$. Then choose a *bijection* $g : S - \{x, y\} \to T - \{t_k\}$, and declare that $f(s) = g(s)$ for any $s \in S - \{x, y\}$. Let's count up these choices. We have that $|S - \{x, y\}| = |T - \{t_k\}| = n - 1$, and we know that the number of bijections between two sets with size $n-1$ is equal to $(n-1)!$. Thus, the total number of onto functions $f : S \to T$ is equal to

$$\binom{n+1}{2} \cdot n \cdot (n-1)! = \binom{n+1}{2} \cdot n!$$

$\square$