HW 12

Problem 0.1 (Chapter 19, problem 1)**.**
For each of the following, say whether f is 1-1 and whether f is onto.

(i) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2 + 2x$ for all $x \in \mathbb{R}$.

(ii) $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x - 2 & \text{if } x > 1 \\ -x & \text{if } -1 \leq x \leq 1 \\ x + 2 & \text{if } x < -1 \end{cases}$$

(iii) $f : \mathbb{Q} \to \mathbb{R}$ defined by $f(x) = (x + \sqrt{2})^2$.

(iv) $f : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ defined by $f(m, n, r) = 2^m 3^n 5^r$ for all $m, n, r \in \mathbb{N}$.

Proof. (i) We note that $x^2 + 2x = x(x + 2)$, so the roots of f are $x = 0$ and $x = -2$. Thus, we have $f(0) = 0$ and $f(-2) = 0$, so f is not 1-1. Graphing f, we get a parabola which opens up and has vertex at $(-1, -1)$. Said another way, f has a global minimum at the point $(-1, -1)$. Thus, $f(x) \geq -1$ for all x, so in particular (for instance) -100 is not equal to $f(x)$ for any $x \in \mathbb{R}$. Therefore, f is not onto.

(ii) We note that $f(2) = 2 - 2 = 0$ (as $2 > 1$, we use the first part of f here), and $f(0) = -0 = 0$ (as $-1 \leq 0 \leq 1$, we use the second part of f here). Therefore, f is not 1-1. We claim that f is onto. To see this, fix $y \in \mathbb{R}$. We consider two cases. If $y \geq 0$, then $y + 2 > 1$, so we use the first part of f to compute $f(y + 2) = (y + 2) - 2 = y$. If $y \leq 0$, then $y - 2 < -1$, so we use the third part of f to compute $f(y - 2) = (y - 2) + 2 = y$. Thus, in all cases we have $y = f(x)$ for some $x \in \mathbb{R}$, and therefore f is onto. Remark: to work this out I started by drawing the graph of f.

(iii) We claim that f is 1-1. To see this, take $x, y \in \mathbb{Q}$, and suppose that $f(x) = f(y)$. Then we have

$$(x + \sqrt{2})^2 = (y + \sqrt{2})^2$$

Let's subtract over to get

$$(x + \sqrt{2})^2 - (y + \sqrt{2})^2 = 0$$

This is a difference of squares, so it factors as

$$(x + \sqrt{2} + y + \sqrt{2})(x + \sqrt{2} - (y + \sqrt{2})) = 0$$

which simplifies to

$$(x + y + 2\sqrt{2})(x - y) = 0.$$

At least one of these two terms has to be zero. We claim that the first term can't be zero. By Proposition 2.4 in the book, because y is rational and $2\sqrt{2}$ is irrational, $y + 2\sqrt{2}$ is also irrational, and hence $-(y + \sqrt{2})$ is irrational. But x is also rational, so $x \neq -(y + 2\sqrt{2})$ and therefore

$$x + y + 2\sqrt{2} \neq 0.$$

Therefore the second term has to be zero. Thus, $x - y = 0$, so $x = y$. This shows that f is 1-1.

We observe that $f(x)$ is always ≥ 0. Thus, f can't be onto. For instance, -100 is not equal to $f(x)$ for any x.

(iv) We claim that f is 1-1. To see this, suppose we have two elements $(m, n, r), (m', n', r') \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ such that

$$f(m, n, r) = f(m', n', r').$$

This means that

$$2^m 3^n 5^r = 2^{m'} 3^{n'} 5^{r'}.$$

By unique factorization (the fundamental theorem of arithmetic!), this implies that $m = m'$, $n = n'$, and $r = r'$. Therefore $(m, n, r) = (m', n', r')$. This shows that f is 1-1.

On the other hand, the image of f consists only of those natural numbers which contain only powers of $2, 3$, and 5 in their prime factorization. Any other number will not be in the image. For instance 7 is not equal to $f(m, n, r)$ for any $(m, n, r) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}$. □

Problem 0.2 (Chapter 19, problem 3)**.**
Two functions $f, g : \mathbb{R} \to \mathbb{R}$ are such that for all $x \in \mathbb{R}$,

$$g(x) = x^2 + x + 3, \qquad \text{and } (g \circ f)(x) = x^2 - 3x + 5.$$

Find the possibilities for f.

Proof. This one is tricky! We can find $f(x)$ using the following method. By the definition of composition, we have

$$(g \circ f)(x) = g(f(x)) = (f(x))^2 + f(x) + 3$$

for all $x \in \mathbb{R}$. On the other hand, we are given that

$$(g \circ f)(x) = x^2 - 3x + 5.$$

Putting these together, we get

$$(f(x))^2 + f(x) + 3 = x^2 - 3x + 5$$

Let's subtract over to get

$$(f(x))^2 + f(x) - x^2 + 3x - 2 = 0$$

We can think about this as a quadratic equation in $f(x)$, with coefficients involving x. That is, this is the quadratic equation

$$ay^2 + by + c = 0$$

where $y = f(x)$, $a = 1$, $b = 1$, and $c = -x^2 + 3x - 2$. The quadratic formula tells us that

$$f(x) = \frac{-1 \pm \sqrt{1 - 4(-x^2 + 3x - 2)}}{2}$$

which simplifies to

$$f(x) = \frac{-1 \pm \sqrt{4x^2 - 12x + 9}}{2} = \frac{-1 \pm \sqrt{(2x-3)^2}}{2} = \frac{-1 \pm (2x-3)}{2}$$

This gives the two possibilities

$$f(x) = x - 2 \qquad \text{and} \qquad f(x) = -x + 1$$

(By the way, at this point we can check our work by pluggin each of these back in to $g(x)$ and making sure we get $(g \circ f)(x) = x^2 - 3x + 5$). □

Problem 0.3 (Chapter 19, problem 6)**.**

(a) Find an onto function from $\mathbb{N}$ to $\mathbb{Z}$.

(b) Find a 1-1 function from $\mathbb{Z}$ to $\mathbb{N}$.

Proof. (a) For both of these, there are lots of functions which work. Here's what I came up with. My thought process for this part is to note that the natural numbers $\mathbb{N}$ are missing all the negative integers (as well as 0). But, I can divide $\mathbb{N}$ up into two pieces consisting of the even and odd numbers, send the even numbers bijectively to $\mathbb{Z}_{>0}$, and send the odd numbers bijectively to $\mathbb{Z}_{\leq 0}$. Define

$$f(x) = \begin{cases} x/2 & \text{if } x \text{ is even} \\ -(x-1)/2 & \text{if } x \text{ is odd} \end{cases}$$

Note that, by the way I wrote the cases, $f(x)$ is an integer for any x. Thus, this defines a function $f : \mathbb{N} \to \mathbb{Z}$. I claim that this function is onto. Indeed, say $y \in \mathbb{Z}$. If $y > 0$, then $y = f(2y)$ (note $2y$ is always even, so we use the first case). If $y \leq 0$, then $y = f(-2y+1)$ (note $-2y+1$ is always odd, so we use the second case). Thus f is onto.

In fact, this f is also 1-1, and therefore a bijection!

(b) Let's use a similar idea as in part (a). I'll think of $\mathbb{Z}$ as being the positive integers $\mathbb{Z}_{>0}$ union the non–positive integers $\mathbb{Z}_{\leq 0}$, and define

$$g(x) = \begin{cases} 2x & \text{if } x > 0 \\ -2x+1 & \text{if } x \leq 0 \end{cases}$$

Actually, this is just the inverse to the function f from part (a). □