HW 11

Problem 0.1 (Chapter 18, problem 1). Which of the following relations are equivalence relations on the given set S?

(i) $S = \mathbb{R}$ and $a \sim b \iff a = b$ or $-b$.

(ii) $S = \mathbb{Z}$ and $a \sim b \iff ab = 0$.

(iii) $S = \mathbb{R}$ and $a \sim b \iff a^2 + a = b^2 + b$.

(iv) S is the set of all people in the world, and $a \sim b$ means a lives within 100 miles of b.

Proof.
(i) This is an equivalence relation.
(ii) This is not an equivalence relation. It fails reflexivity, because for instance $1 \cdot 1 \neq 0$, so $1 \nsim 1$. It does satisfy symmetry. It also fails transitivity. For instance, we have $1 \cdot 0 = 0$, so $1 \sim 0$, and we have $0 \cdot 2 = 0$, so $0 \sim 2$, but $1 \cdot 2 \neq 0$, so $1 \nsim 2$.
(iii) This is an equivalence relation.
(iv) This is not an equivalence relation. It satisfies reflexivity and symmetry. But, it fails transitivity. This is because we can have three people a, b, c such that a lives within 100 miles of b, and b lives within 100 miles of c, but a doesn't live within 100 miles of c (imagine the case when they are spaced out along a straight line). $\square$

Problem 0.2 (Chapter 18, problem 5).

1. How many relations are there on the set $\{1, 2\}$?
2. How many relations are there on the set $\{1, 2, 3\}$ that are both reflexive and symmetric?
3. How many relations are there on the set $\{1, 2, \ldots, n\}$?

Proof. (1) A relation on a set S is (by definition) a subset of $S \times S$. In our case, $\{1, 2\} \times \{1, 2\}$ has four elements, so there are $2^4 = 16$ possible relations.
(2) Let $\sim$ be a relation which is reflexive and symmetric. There are $3^2 = 9$ total pairs of elements of $\{1, 2, 3\}$ which could be in the relation. To be reflexive means that $1 \sim 1$, $2 \sim 2$, and $3 \sim 3$. Of the six remaining pairs, the symmetry condition implies that

$$\begin{aligned} 1 \sim 2 &\iff 2 \sim 1 \\ 1 \sim 3 &\iff 3 \sim 1 \\ 2 \sim 3 &\iff 3 \sim 2 \end{aligned}$$

Given any subset of the left hand pairs $1 \sim 2$, $1 \sim 3$, and $2 \sim 3$, we get a reflexive and symmetric relation by including the mirror image pairs and all the identity pairs $1 \sim 1$, $2 \sim 2$, and $3 \sim 3$. So, the number of such relations is equal to the number of subsets of a set with 3 elements, which is $2^3 = 8$.
(3) As in part (1), a relation on $\{1, 2, \ldots, n\}$ is a subset of the product $\{1, 2, \ldots, n\} \times \{1, 2, \ldots, n\}$. This has n^2 elements, so there are $2^{(n^2)}$ relations. $\square$

Problem 0.3 (Chapter 18, problem 6)**.** Let $S = \{1, 2, 3, 4\}$ and suppose that $\sim$ is an equivalence relation on S. You are given the information that $1 \sim 2$ and $2 \sim 3$. Show that there are exactly two possibilities for the relation $\sim$, and describe both.

Proof. By Proposition 18.1 (plus a little), given an equivalence relation on a set S, the set of equivalence classes gives a partition of S. Furthermore, this partition completely determines the equivalence relation, because we have $a \sim b$ if and only if a and b are in the same set in the partition. Thus, this problem can be rephrased as the problem of describing all partitions of $\{1, 2, 3, 4\}$ such that 1 and 2 are in the same subset and 2 and 3 are in the same subset. These two combined shows that $1, 2$, and 3 are all in the same subset. There are only two ways to complete this to a partition: we can either include 4 in the same subset as well, giving the partition

$$S_1 = \{1, 2, 3, 4\}$$

or we can put 4 into its own subset, giving the partition

$$S_1 = \{1, 2, 3\} \qquad S_2 = \{4\}$$

The corresponding equivalence relations on $\{1, 2, 3, 4\}$ are as described above: we have $a \sim b$ if and only if a and b are in the same subset of the partition. $\square$