HW 10

March 30, 2023

Problem 0.1 (Chapter 17, problem 1)**.**

(a) Let A, B be sets. Prove that $A \cup B = A$ if and only if $B \subset A$.

(b) Prove that $(A - C) \cap (B - C) = (A \cap B) - C$ for all sets A, B, C.

Proof. (a) Suppose that $A \cup B = A$. We then have $B \subset A \cup B = A$, so $B \subset A$. Suppose that $B \subset A$. Say $x \in A \cup B$. Then either $x \in A$ or $x \in B$. In the second case, we have $x \in B \subset A$. So either way, $x \in A$, and therefore $A \cup B \subset A$. On the other hand, we always have $A \subset A \cup B$. We conclude that $A \cup B = A$.
(b)

$$\begin{aligned} x \in (A - C) \cap (B - C) &\iff x \in A - C \text{ and } x \in B - C \\ &\iff x \in A \text{ and } x \in B \text{ and } x \notin C \\ &\iff x \in (A \cap B) - C \end{aligned}$$

□

Problem 0.2 (Chapter 17, problem 5)**.** How many integers are there between 1000 and 9999 that contain the digits $0, 8$, and 9 at least once each?

Proof. I thought this one was a bit tricky! Here is the most elegant way I found to do this. Consider the following sets: we let A be the set of integers between 1000 and 9999 which do *not* contain 0 as a digit, let B be the set which do *not* contain 8, and let C be the set which do *not* contain 9. Why is this helpful? Well, the first thing to notice is that $A \cup B \cup C$ is the set of integers between 1000 and 9999 which do not contain at least one of the digits $0, 8$, and 9. Thus, the *complement* of $A \cup B \cup C$ is the set of integers which contain $0, 8$, and 9 at least once each. There are 9000 numbers between 1000 and 9999, so the answer to the question is

$$9000 - |A \cup B \cup C|$$

This is only helpful though if we can compute $|A \cup B \cup C|$. The second thing to notice is that we can do this with the inclusion–exclusion principle, which says that

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

Crucially, we will see that all of the right hand terms can be computed pretty quickly. So, the plan is to compute each of these, add them up with the right signs to obtain $|A \cup B \cup C|$, and then subtract this from 9000 to get the answer to the question.

Let's start with A. For a number in A, there are 9 choices for each of the four digits, so we have

$$|A| = 9^4 = 6561,$$

We have

$$|B| = 8 \cdot 9^3 = 5832,$$

because the first digit of a number in B can be any number besides 0 or 8, while the other digits can be anything besides 8. For the exact same reason, we have

$$|C| = 8 \cdot 9^3 = 5832.$$

Next let's compute $|A \cap B|$. This is the set of integers between 1000 and 9999 which contain neither 0 nor 8 as a digit. There are 8 choices for each digit of a number in $A \cap B$, so

$$|A \cap B| = 8^4 = 4096.$$

For the same reason, we have

$$|A \cap C| = 8^4 = 4096.$$

We have

$$|B \cap C| = 7 \cdot 8^3 = 3584$$

because, for a number in $B \cap C$, there are 7 choices for the first digit and 8 choices for the other three digits (again, because the first digit can't be a 0). Finally, we have that $A \cap B \cap C$ is the set of integers between 1000 and 9999 which contain none of $0, 8$, or 9 as a digit. For a number in $A \cap B \cap C$, there are 7 possibilities for each digit, so we have

$$|A \cap B \cap C| = 7^4 = 2401.$$

Putting this all together, we get

$$\begin{aligned} |A \cup B \cup C| &= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \\ &= 6561 + 5832 + 5832 - 4096 - 4096 - 3584 + 2401 \\ &= 8850 \end{aligned}$$

Therefore the answer to the question is

$$9000 - 8850 = 150$$

$\square$

Problem 0.3 (Chapter 17, problem 6)**.** How many integers between 1 and 10000 are neither squares nor cubes?

Proof. Let S be the set of integers between 1 and 10000 which are squares, and let C be the set which are cubes. We will find $|S|$, $|T|$, and $S \cap T$. To find S, we note that $100^2 = 10000$. Thus, we have

$$S = \left\{1^2, 2^2, 3^2, \ldots, 100^2\right\}$$

so $|S| = 100$. To find T, a bit of trial and error shows that $21^2 = 9261$ and $22^2 = 10648$. Thus, 21^3 is the biggest cube which is less than 10000, and so we have

$$T = \left\{1^3, 2^3, 3^3, \ldots, 21^3\right\}$$

and therefore $|T| = 21$. Finally we compute $|S \cap T|$. Note that $S \cap T$ is the set of integers between 1 and 10000 which are sixth powers. Thus we have

$$S \cap T = \left\{1^6, 2^6, 3^6, 4^6 \ldots\right\}$$

and so $|S \cap T| = 4$. By the inclusion–exclusion principle, we get

$$\begin{aligned} |S \cup T| &= |S| + |T| - |S \cap T| \\ &= 100 + 21 - 4 \\ &= 117 \end{aligned}$$

This is the number of integers between 1 and 10000 which are either squares or cubes. Thus, the answer to the question is $10000 - 117 = 9883$. $\square$