6440/5470 Midterm 2. Partial Solution

1) $\dot{r} = r(1+\mu r + r^2)$

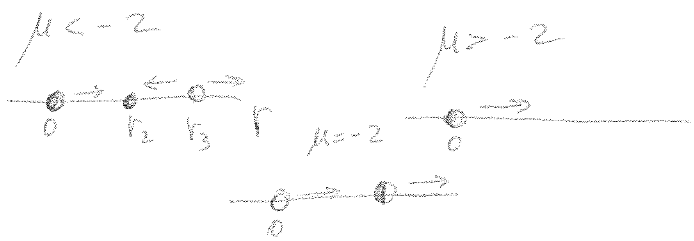
$\dot{\theta} = 1 + \frac{1}{2}\sin\theta$

First, discuss $\theta$ dynamics $\dot{\theta} > 0$, so $\theta$ always moves in the counterclockwise direction, ~~~~ but the speed of rotation depends on the angle.

For $r$: $\dot{r} = r(1+\mu r + r^2)$

Find f. points: $r^* = 0$ or $r^2 + \mu r + 1 = 0$

$$r = \frac{-\mu \pm \sqrt{\mu^2 - 4}}{2},$$



so for $|\mu| \le 2$, $r^* = 0$;

for $\mu > 2 \Rightarrow r^* = 0$ (the others are negative);

for $\mu < -2 \Rightarrow r_1^* = 0$, $r_{2,3}^* = \frac{-\mu \pm \sqrt{\mu^2 - 4}}{2}$

Critical value of $\mu$: $\underline{\mu_c = -2}$. At $\mu_c$ $r_{2,3}^* = 1$.

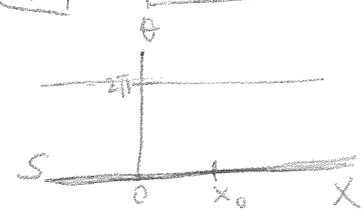
In the phase plane for $\mu < -2$  a fixed point (unstable spiral) and two cycles

as $\mu \to -2^-$ the cycles collide and disappear in a $\boxed{\text{"saddle-node of cycles"}}$ bifurcation.

2) $\boxed{\begin{aligned} \dot{x} &= \mu x \\ \dot{\theta} &= \omega,\ \omega > 0. \end{aligned}}$ a) Take the cut $S: \theta = 0$.

Poincaré map: Pick $x_0$. The flow will go increasing in $\theta$, to return to $S$ at $t = \frac{2\pi}{\omega}$.

To find the image $x_1$: $\frac{dx}{dt} = \mu x$.

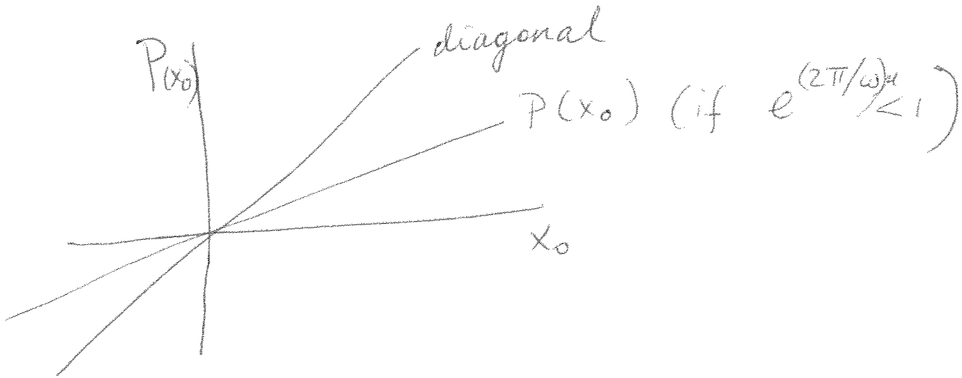


For $x \ne 0$: $\frac{dx}{\mu x} = dt$. Over the trajectory $\int_{x_0}^{x_1} \frac{dx}{\mu x} = \int_0^{2\pi/\omega} dt$.

$\frac{1}{\mu} \ln|x| \Big|_{x_0}^{x_1} = \frac{2\pi}{\omega}$

$\frac{1}{\mu}\ln|x_1| - \frac{1}{\mu}\ln|x_0| = \frac{2\pi}{\omega}$ $\quad |x_1| = e^{\frac{2\pi}{\omega}\mu} \cdot |x_0|$ with $x_1 \ge 0$ if $x_0 \ge 0$ and $x_1 < 0$ if $x_0 < 0$

We found $P(x_0) = x_1 = x_0\, e^{\frac{2\pi}{\omega}\mu}$.




b) $\boxed{P(0) = 0}$, so after a rotation with $x(0) = x_0 = 0$ it would come to the same point, corresponding to a periodic orbit in the original system.

c) The point $x_0 = 0$ is a fixed point of the map $P$. If it is stable, the periodic orbit would be stable. It happens when $0 < P'(0) < 1$.

$P'(0) = e^{\frac{2\pi}{\omega}\mu}$. $e^{\frac{2\pi}{\omega}\mu} < 1$ when $\mu < 0$.

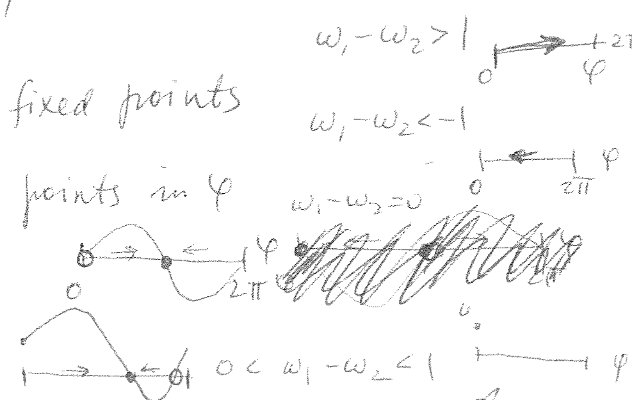
$$\boxed{\begin{array}{l} \mu < 0 \text{ stable periodic orbit} \\ \mu > 0 \text{ unstable periodic orbit} \end{array}}$$

3) $\frac{d}{dt}(\theta_1 - \theta_2) = \omega_1 - \omega_2 + \sin(\theta_1 - \theta_2)$

Let $\varphi = \theta_1 - \theta_2$. $\dot{\varphi} = (\omega_1 - \omega_2) + \sin\varphi$

$|\omega_1 - \omega_2| > 1 \Rightarrow$ no $\varphi$-fixed points

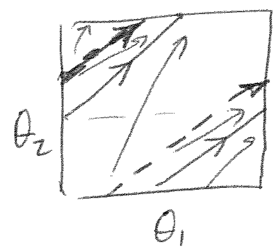
$|\omega_1 - \omega_2| < 1 \Rightarrow$ 2 fixed points in $\varphi$



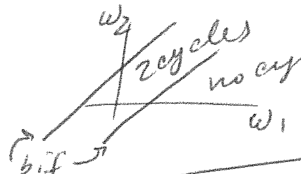

bifurcations: saddle-node at $\omega_1 - \omega_2 = \pm 1$.

Fixed point in $\varphi$ means that $\theta_2(t)-\theta_1(t) = \text{const}$ ~~is a constant~~, so lines $\theta_2 = \theta_1 + \varphi^*$ are invariant under this system. We have not had an example like this in class, so I did not deduct points for simply saying that fixed points for $\varphi^*$ correspond to stable and unstable cycles in $(\theta_1, \theta_2)$.

For example, for $|\omega_1 - \omega_2| < 1$ there are $\varphi_1^*$ (stable) and $\varphi_2^*$ (unstable), so $\theta_2 = \theta_1 + \varphi_1^*$ - stable, $\theta_2 = \theta_1 + \varphi_2^*$ unstable.




In this case part b) will look like this: $|\omega_1 - \omega_2| < 1$ - 2 periodic orbits




Bifurcations (saddle-node of cycles) at $\omega_2 = \omega_1 \pm 1$

For ~~......~~ the solution to be correct completely, one would need to consider a system

$$\begin{cases} \dot{\varphi} = \omega_1 - \omega_2 + \sin\varphi \\ \dot{\psi} = \omega_1 + \omega_2 + \cos\psi \end{cases}, \text{ where } \varphi = \theta_1 - \theta_2, \ \psi = \theta_1 + \theta_2$$

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(4). $\ddot{x} + k(x^2 - y)\dot{x} + x = 1, \quad k \gg 1$

a) $w = \dot{x} + kF(x)$

$\Rightarrow \quad \dot{x} = w - kF(x)$

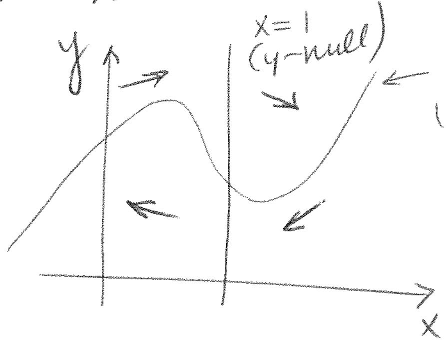
$\dot{w} = \ddot{x} + kF'(x)\cdot\dot{x} = \ddot{x} + k(x^2 - y)\dot{x} = 1 - x$

from original eq.

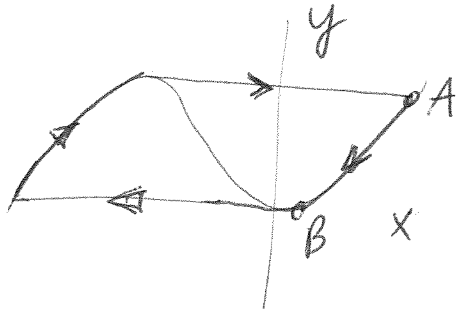
$y = \frac{w}{k}$, so $\dot{y} = \frac{1}{k}\dot{w}$ or

$$\begin{cases} \dot{x} = k(y - F(x)) \\ \dot{y} = \dfrac{1-x}{k} \end{cases}$$

b) Plot nullclines:




Add direction arrows.
Notice that away from $y = F(x)$ $\dot{x} \gg \dot{y}$, so the flow is nearly horizontal.




On $y = F(x)$ the flow is slow and dominated by

$\dot{y} = \frac{1-x}{k}$

$y = F(x)$.

c) $T \approx 2\int_{t_A}^{t_B} dt \; \cancel{= 2\int_{x_A}^{x_B} dx}$ (A & B marked above)

$$\frac{dy}{dt} = \frac{1-x}{k}, \quad \frac{dy}{dt} = F'(x)\frac{dx}{dt} \;\Rightarrow\; \frac{dx}{dt}\cdot(x^2 - y) = \frac{1-x}{k} \qquad dx \cdot \frac{(x^2-y)k}{1-x} = dt$$

$T \approx 2k \int_{x_A}^{x_B} \frac{x^2-4}{1-x}\,dx$, where $x_B$ is the point of minimum and $x_A$ is such that $F(x_A) = y_{max}$, where $y_{max}$ is the value at the point of maximum.

5) $\dot{x} = x^2 - x^3 - xy$
$\dot{y} = xy - by$

a) Jacobian: $\begin{pmatrix} 2x - 3x^2 - y & -x \\ y & x - b \end{pmatrix}$.

at $(b, b-b^2)$: $\begin{pmatrix} 2b - 3b^2 - b + b^2 & -b \\ b - b^2 & 0 \end{pmatrix} = \begin{pmatrix} b - 2b^2 & -b \\ b - b^2 & 0 \end{pmatrix}$.

$\tau = b - 2b^2$

$\Delta = b^2 - b^3$. at $b = \frac{1}{2}$: $\left.\begin{array}{l} \tau = 0 \\ \Delta = \frac{1}{4} - \frac{1}{8} > 0 \end{array}\right\}$ Hopf. bif.

b). $T \approx \frac{2\pi}{\operatorname{Im}\lambda}$ or frequency $\omega = \operatorname{Im}\lambda$.

Jacobian at $b = \frac{1}{2}$: $\begin{pmatrix} 0 & -\frac{1}{2} \\ \frac{1}{4} & 0 \end{pmatrix}$. $\begin{vmatrix} -\lambda & -\frac{1}{2} \\ \frac{1}{4} & -\lambda \end{vmatrix} = \lambda^2 + \frac{1}{8} = 0$

$\lambda = \pm i \frac{1}{2\sqrt{2}}$

$\boxed{\omega = \operatorname{Im}\lambda = \frac{1}{2\sqrt{2}}}$

c). The main feature: unstable point and stable cycle around it (from Hopf) + nullclines, direction arrows.