5470/6440
Spring 2017

Midterm 1

(1) $\dot{x} = 4x - 5x^2 + x^3$

Fixed points: $4x - 5x^2 + x^3 = 0$

$$x(4 - 5x + x^2) = 0$$

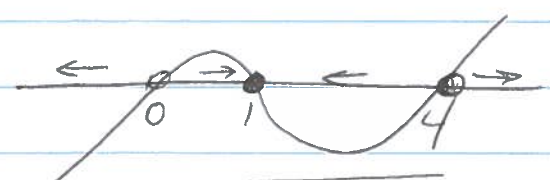
$$\underline{x^* = 0} \quad \text{or} \quad \underline{x^* = 4} \quad \text{or} \quad \underline{x^* = 1}$$

Stability: $f(x) = 4x - 5x^2 + x^3$

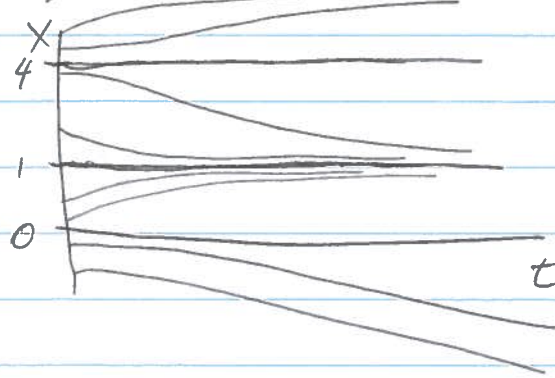
$$f'(x) = 4 - 10x + 3x^2$$

$f'(0) = 4 > 0$, so $x^* = 0$ – unstable
$f'(1) = -3 < 0$, so $x^* = +1$ – stable
$f'(4) = 12 > 0$, so $x^* = 4$ – unstable

Phase line:

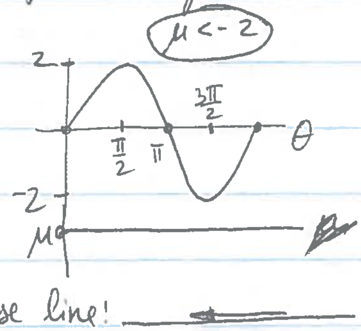


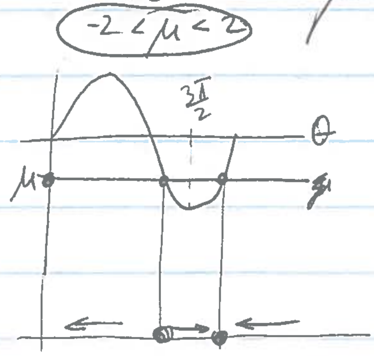
Time courses:

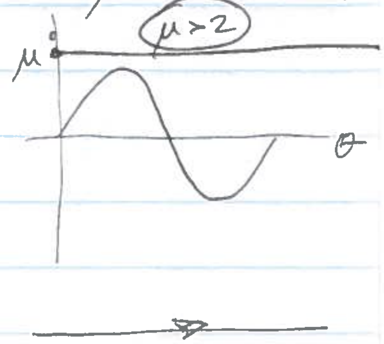


② $\dot{\theta} = \mu - 2\sin\theta$

a) fixed points: where $2\sin\theta = \mu$. When $\mu > 2\sin\theta, f(\theta) > 0$ $\longrightarrow$

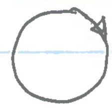


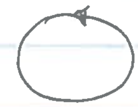




→ phase line:

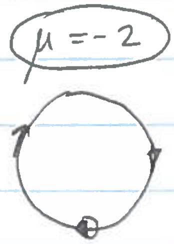
→ circle phase space:

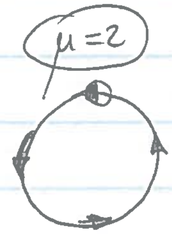




→ negative direction in θ is clockwise

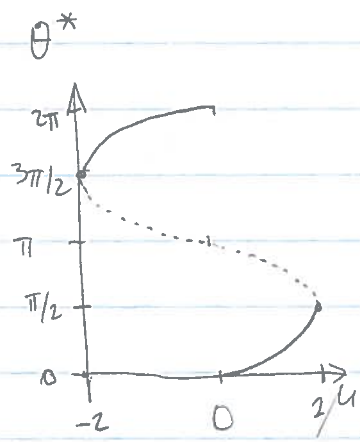
As μ grows the points move further apart, but then closer as $\mu \to 2$



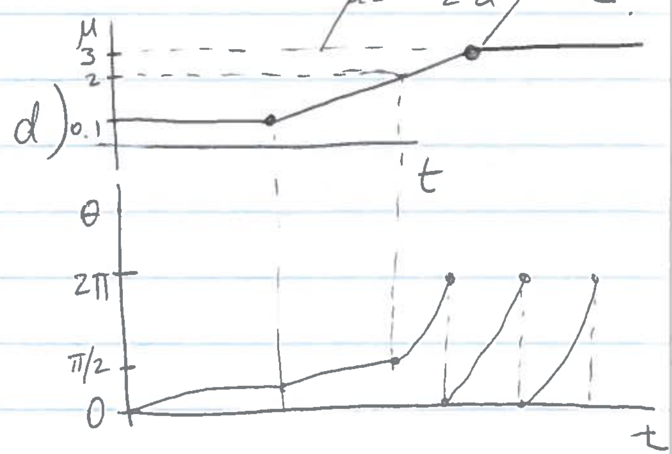


b) bifurcations are saddle-node at $\mu = -2$ & $\mu = 2$.

c)



d)



④ $\dot{x} = x(3-2x-2y)$
$\dot{y} = y(2-x-y)$

a) Fixed points: $\begin{cases} x(3-2x-2y)=0 \\ y(2-x-y)=0 \end{cases}$

$\begin{cases} x=0 \quad \text{or} \quad y=\frac{3}{2}-x \\ y(2-x-y)=0 \end{cases}$

$\begin{cases} x=0 \\ y=0 \end{cases}$ $\begin{cases} x=0 \\ y=2 \end{cases}$ $\begin{cases} x=\frac{3}{2} \\ y=0 \end{cases}$ $\left.\begin{cases} y=\frac{3}{2}-x \\ 2-x-\frac{3}{2}+x=0 \end{cases}\right\}$ no solution.

So, fixed points $(0,0)$, $(0,2)$, $\left(\frac{3}{2},0\right)$.

Classify them:

Jacobian: $J=\begin{pmatrix} 3-4x-2y & -2x \\ -y & 2-x-2y \end{pmatrix}$.

$J\big|_{(0,0)} = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$. $\lambda_1=3, \lambda_2=2$. $(0,0)$ - unstable node.

$J\big|_{(0,2)} = \begin{pmatrix} -1 & 0 \\ -2 & -2 \end{pmatrix}$ $\lambda_1=-1, \lambda_2=-2$ $(0,2)$ - stable node

$J\big|_{(\frac{3}{2},0)} = \begin{pmatrix} -3 & -3 \\ 0 & \frac{1}{2} \end{pmatrix}$ $\lambda_1=-3, \lambda_2=\frac{1}{2}$ $(3/2,0)$ - saddle.

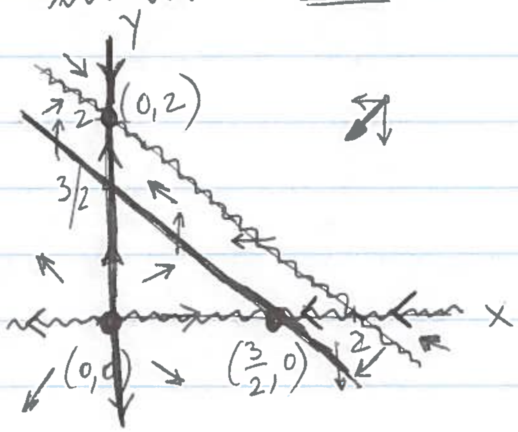
b) Nullclines & the vector field.

x-null.: $x=0$ or $3-2x-2y=0$

$y=\frac{3}{2}-x$

y-null.: $y=0$ or $2-x-y=0$

$y=2-x$



c) Phase portrait.

5 $\ddot{x} = x - x^2$

a) equilibrium points

$$\begin{cases} \dot{x} = y \\ \dot{y} = x - x^2 \end{cases} \qquad \begin{cases} y = 0 \\ x(1-x) = 0 \end{cases} \qquad \boxed{(0,0) \quad (1,0)}$$

Jacobian: $\begin{pmatrix} 0 & 1 \\ 1-2x & 0 \end{pmatrix}$

at $(0,0)$: $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$; $\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 1 = 0$

$\lambda_{1,2} = \pm 1$ <u>$(0,0)$-saddle</u>

at $(1,0)$: $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$; $\begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0$

$\lambda_{1,2} = \pm i$ <u>$(1,0)$-center</u>
in the linear
classification

b) conserved quantity.
From class, we can check $E = \frac{m\dot{x}^2}{2} + V(x)$, where the original system is
$m\ddot{x} = F(x)$, $\frac{dV}{dx} = -F(x)$.

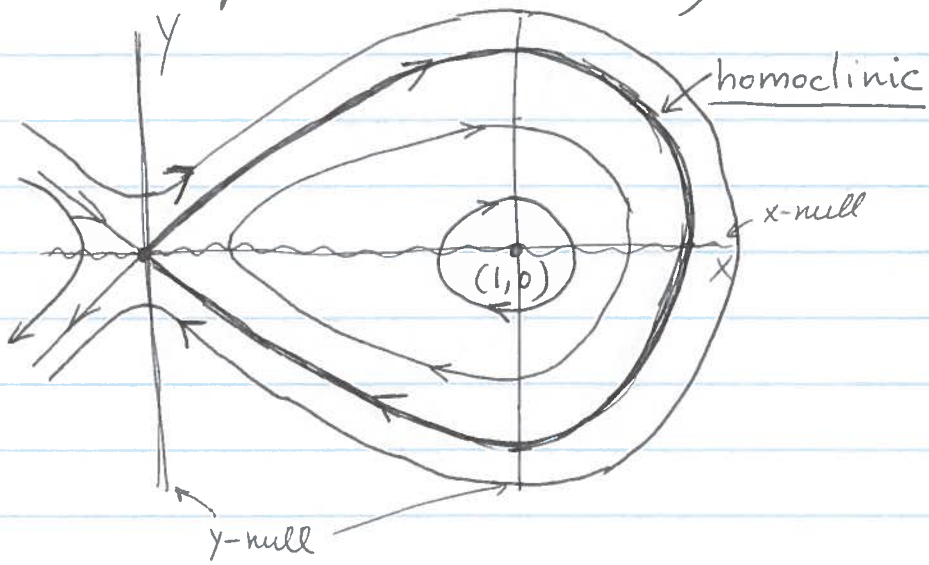
$m=1$, $\frac{dV}{dx} = -x + x^2$; $-\frac{x^2}{2} + \frac{x^3}{3} = V(x)$.

$$\boxed{E = \frac{\dot{x}^2}{2} - \frac{x^2}{2} + \frac{x^3}{3}.}$$

Check: $\frac{dE}{dt} = \frac{2}{2} \cdot \dot{x}\ddot{x} - x\dot{x} + x^2\dot{x} = \dot{x}(\ddot{x} - x + x^2)$

$= \dot{x}(x - x^2 - x + x^2) = 0$, so <u>$E = \text{const}$</u>

c) To plot: the system is conservative, so linear center will probably stay a center. It is also reversible (or there is a symmetry with respect to x-axis)



d) Solutions inside the homoclinic are periodic closed orbits, and those that are outside are not.

(6) $\dot{x} = \sin y$
$\dot{y} = \sin x$

a) Reversibility: $t \to (-t)$, and $y \to (-y)$ preserves the system:

$$\frac{dx}{d(-t)} = -\frac{dx}{dt} = -\sin y = \sin y$$

$$\frac{d(-y)}{d(-t)} = \frac{dy}{dt} = \sin x$$

b) F. points:

$$\begin{cases} \sin y = 0 \\ \sin x = 0 \end{cases} \quad \begin{cases} x = \pi k \\ y = \pi n \end{cases}, \quad \forall k, n \in \mathbb{Z}.$$

Classify: $\begin{pmatrix} 0 & \cos y \\ \cos x & 0 \end{pmatrix}$ $\quad \lambda^2 = \cos x \cdot \cos y$.

If $\cos x^*$ & $\cos y^*$ have the same sign - saddle; otherwise - center.

c) Start with $x(0) = y(0)$. Then $\dot{x} = \sin y$
$\dot{y} = \sin y$.

Since the derivatives are equal, we will continue to have $x(t) = y(t)$.

Same for $y = -x$. $\dot{x} = \sin y$, $\dot{y} = \sin(-y) = -\sin y = -\dot{x}$, so it will stay on the line.

d)