Homework on $Out(F_n)$

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Problems labeled with an asterisk are more difficult/technical and constitute the take-home final.

1 Folding and applications

H is a finitely generated subgroup of F_n.

1. Find a basis of the subgroup

$$H = \langle b\bar{a}baaba, ab\bar{a}baba, ab\bar{a}b\bar{a}\bar{b}a\bar{b}\rangle < \langle a, b\rangle$$

2. Given $w \in F_n$ give an algorithm to decide whether $w \in H$. E.g. show $a \notin H$ of #1.

3. Given $w \in F_n$ give an algorithm to decide whether w is conjugate into H.

4. Can you tell if H is normal in F_n?

5. Can you tell if H has finite index in F_n?

6. Suppose H is a finitely generated normal subgroup of F_n. Show that either H has finite index in F_n or $H = \{1\}$.

7. Given a homomorphism $h : F_n \to F_m$, can you tell if h is injective, surjective, bijective? Answer: Injective iff there are no folds of the second kind. Surjective iff the last map is a homeomorphism. In particular, show that F_n is *hopfian*, i.e. every epimorphism $F_n \to F_n$ is an automorphism.

8. Let $h : \langle a, b\rangle \to \langle a, b\rangle$ be given by $h(a) = abbab$, $h(b) = bababbab$. Show that h is an automorphism and compute h^{-1}. (You can do this by messing about. But try to do it algorithmically, that is, decompose h into a product of Nielsen generators and then compose the inverses in opposite order. The point is that this can be programmed on a computer.)

9. Show that for every homomorphism $h : F_n \to F_m$ there is a free factorization $F_n = A * B$ such that h kills A and is injective on B.

10. Show that for every finitely generated $H \subset F_n$ there is a subgroup $H' \subset F_n$ such that $H \subset H'$, H is a free factor in H', and H' has finite index in F_n. This is called Marshall Hall's theorem. You can find H' algorithmically. Do it for H in the example from #1. Hint: Add some edges to G to turn an immersion $G \to Y$ into a covering map.

11. Can you always compute the normalizer

$$N(H) = \{\gamma \in F_n \mid \gamma H \gamma^{-1} = H\}?$$

What can you say about the index $[N(H) : H]$? (Answer: it is always finite and bounded by the number of vertices in the graph representing H. Recall that $N(H)/H$ is the deck group.) E.g. show that $N(H) = H$ for H as in #1.

12. If T and T' are two maximal trees, show that there is a sequence $T = T_0, T_1, \cdots, T_k = T'$ of maximal trees such that any two consecutive trees differ in only one edge, as in the lecture.

13.* This is a bit more ambitious. Consider the simplicial complex whose vertices are non-closed edges of G, and a collection of edges spans a simplex if their union is a forest. Draw some examples. Can you make a conjecture about the homotopy type of the complex?

14.* Read the wonderful paper *Topology of finite graphs* by John Stallings (Inventiones 71 (1983) 551-565.)