COUNTING MAPS FROM A SURFACE TO A GRAPH

MLADEN BESTVINA AND MARK FEIGHN

Contents

1. Introduction and results

Fix a nonabelian free group $\mathbb{F}$ of finite rank and let G be a finitely generated (or f.g. for short) group with a f.g. subgroup P. In his work on the Tarski problem, Zlil Sela considers the following question. In how many ways can a given homomorphism $P \to \mathbb{F}$ be extended to G? Of course without further restrictions the answer is often infinitely many. He goes on to define a natural equivalence relation on the set of extensions (described below in our setting) and obtains the remarkable result:

Theorem 1.1 (Sela [5]). *Suppose that G is freely indecomposable rel P. There is a finite set $\{q_i : G \to G_i\}$ of proper quotients and a number $f = f(G, P)$ so that each homomorphism $P \to \mathbb{F}$ has at most f equivalence classes of extensions to G with the property that no element of the equivalence class factors through some q_i.*

Not much was known about $f(G, P)$. For example, Sela asked whether there was a sequence of examples (G_i, P_i) with $\lim f(G_i, P_i) = \infty$. Our main result is to show that there is such a sequence. In fact, in our sequence G_i will be the fundamental group of an orientable surface of genus i with P_i representing its boundary and we show that $f(G_i, P_i) \geq 2^i$.

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We now describe our results in more detail. For x in the commutator subgroup $[\mathbb{F},\mathbb{F}]$ of $\mathbb{F}$ define the *algebraic genus of* x denoted

$$\mathsf{a\text{-}genus}\, x$$

as the smallest $g \geq 0$ such that x is the product

$$x = [p_1, q_1] \cdots [p_g, q_g]$$

of g commutators. Of course, $\mathsf{a\text{-}genus}\, x$ depends only on the conjugacy class $[[x]]$ of x in $\mathbb{F}$ and we define $\mathsf{a\text{-}genus}\,[[x]] := \mathsf{a\text{-}genus}\, x$. Topologically, we can represent the situation by mapping an orientable surface $\mathbb{S}_g$ of genus g and one boundary component to a graph representing $\mathbb{F}$. Also define

$$\mathsf{num}\, x$$

to be the maximal number of different ways in which $x \in \mathbb{F}$ with algebraic genus g can be written as the product of g commutators. By "different" we mean inequivalent under the relation "$\sim$" that we now define.

Identify $\mathbb{F}_{2g} = \langle a_1, b_1, \cdots, a_g, b_g \rangle$ with the fundamental group of $\mathbb{S}_g$ and set $\partial_g = [a_1, b_1] \cdots [a_g, b_g]$. For $x \in \mathbb{F}$ a *representation of algebraic genus g for x* is a homomorphism $\psi \in \mathsf{Hom}(\mathbb{F}_{2g}, \mathbb{F})$ such that $\psi(\partial_g) = x$. The equivalence relation "$\sim$" on representations is generated by

(1) $\psi \circ \theta \sim \psi$ where $\theta \in \mathsf{Aut}(\mathbb{F}_{2g})$ satisfies $\theta(\partial_g) = \partial_g$.
(2) $\psi \circ i_{\partial^n} \sim \psi$, $n \in \mathbb{Z}$, where i_{∂^n} is conjugation by ∂^n.
(3) $\theta \sim \psi$ where θ is a fractional Dehn twist[1] of ψ.
(4) $i_z \circ \psi \sim \psi$ where z or a root of x.

Example 1.2. Let $\mathbb{F} = \langle u, v \rangle$, $\mathbb{F}_2 = \langle a_1, b_2 \rangle$, and $x_m = [u^m, v]$. For $m, n \in \mathbb{Z}$, the homomorphism $\phi_{m,n} : \mathbb{F}_2 \to \mathbb{F}$ given by $a_1 \mapsto u^m$ and $b_1 \mapsto uv^n$ is a representation for x_m. The homomorphisms $\phi_{m,n}$ and $\phi_{m,n'}$ are equivalent by a partial Dehn twist whereas they are equivalent by a Dehn twist iff $n \equiv n' \mod m$. This shows the need for including (3).

Remark 1.3. By (1) and (2), the group of outer automorphisms preserving the conjugacy class of ∂_g acts on equivalence classes of representations. This in turn may be identified with the modular group $\mathsf{Mod}(\mathbb{S}_g)$, see [7]. $\mathsf{Mod}(\mathbb{S}_g)$ is generated by Dehn twists [1, 2]. So, we could have defined "$\sim$" using only (3) and (4). In fact, (4) can be interpreted as a fractional Dehn twist in the the boundary curve.

[1] A homomorphism θ is a *fractional Dehn twist in σ of ψ* if there is a simple closed curve σ on $\mathbb{S}_g$ such that θ is given as follows. If σ induces the splitting $\mathbb{F}_{2g} = A *_C B$ and if $z \in \mathbb{F}$ centralizes $\phi(\partial_g)$ then $\theta|A = \psi|A$ and $\theta|B = i_z \circ (\psi|B)$. If $\mathbb{F}_{2g} = A *_C$ then $\theta|A = \psi|A$ and $\theta(t) = \psi(t)z$ where t is the stable letter.

Finally, define

$$f_{\mathbb{F}}(g) = \sup\{\mathsf{num}\, x \mid \mathsf{a\text{-}genus}\, x = g\}.$$

That $f_{\mathbb{F}}(g)$ is finite is a consequence of Theorem 1.1. In Corollary 4.7 we show that $f_{\mathbb{F}}$ is independent of $\mathbb{F}$.

It is not hard to see that a "generic" element of algebraic genus 1 can be written as $[p, q]$ in essentially only one way (up to the above operations). However, it should also be reasonable to expect that $f_{\mathbb{F}}(1) > 1$ – take a "generic" map from the genus 2 surface to $\mathbb{F}$, and then the image x of the waist curve is written as $[p, q]$ in two ways. It takes a little bit of work to show that they really are different. This is the content of Section 3. This reproduces a result of Lyndon and Wicks [3][2].

For higher genera this conceptual argument fails to show $f_{\mathbb{F}}(g) > 2$. The reason is that we do not know explicitly the MR-diagram[3] for the group obtained by gluing say 3 surfaces with boundary along their boundaries. The only "obvious" quotients are obtained by identifying two of the surfaces or killing the common boundary. To find interesting examples one would have to show that there are other maximal limit group quotients of this group.

However, we will argue that $f_{\mathbb{F}}(g) \geq 2^g$. This is the content of Section 4. For example, to see $f_{\mathbb{F}}(2) \geq 4$ we form the "boundary connected sum" of genus 1 examples. Each piece bounds in two ways, so we expect the sum to bound in 4 ways.

In order to deal with fractional Dehn twists it is convenient to consider more restrictive products of commutators.

Definition 1.4. Say an injective representation $\psi : \mathbb{F}_{2g} \to \mathbb{F}$ given by

$$x = [p_1, q_1] \cdots [p_g, q_g]$$

of element x with algebraic genus g is *admissible* if the group

$$\mathsf{Im}\, \psi = \langle p_1, q_1, \cdots, p_g, q_g \rangle$$

is a primitive[4] subgroup of $\mathbb{F}$.

Proposition 1.5. *Let ψ be an admissible representation for x and suppose $\theta \sim \psi$. Then θ is also admissible and $\mathsf{Im}\, \theta$ is conjugate to $\mathsf{Im}\, \psi$.*

[2]Thanks to Leo Comerford for pointing us to this article.

[3]Some comments are meant for those familiar with Sela's work on the Tarski problems. The theorems and proofs in this paper do not depend on such a familiarity.

[4]closed under taking roots, *root-closed* in [3]

Proof. It is clear that the modular group operations and conjugations preserve the conjugacy class of $\mathsf{Im}\,\psi$. In the presence of primitivity, simple closed curves represent indivisible[5] elements of $\mathbb{F}$ and hence fractional Dehn twists are Dehn twists. □

Definition 1.6. For a subgroup H of $\mathbb{F}$, $[H]$ denotes its conjugacy class. Define

$$\mathsf{num}'\,x := \left|\{[\mathsf{Im}\,\psi] : \psi \text{ is an admissible representation for } x\}\right|$$

and

$$f'_{\mathbb{F}}(g) = \sup\{\mathsf{num}'\,x \mid \mathsf{a\text{-}genus}\,x = g\}$$

We then have

$$f_{\mathbb{F}}(g) \geq f'_{\mathbb{F}}(g)$$

We will see that $f'_{\mathbb{F}}$ is also independent of $\mathbb{F}$. Our main theorem is:

Theorem 1.7. *$f'_{\mathbb{F}}(1) \geq 2$ and $f'_{\mathbb{F}}(m+n) \geq f'_{\mathbb{F}}(m) f'_{\mathbb{F}}(n)$.*

Corollary 1.8. $f_{\mathbb{F}}(g) \geq f'_{\mathbb{F}}(g) \geq 2^g$

2. Labeled graphs and geometric genus

$\mathbb{F}$ is a non-abelian free group with fixed finite basis $\mathcal{B}$. The cyclic word obtained by cyclically reducing the $\mathcal{B}$-word w is denoted $[[w]]$. There is a 1-1 correspondence between cyclically reduced cyclic $\mathcal{B}$-words and conjugacy classes of elements of $\mathbb{F}$. If $x \in \mathbb{F}$, then $[[x]]$ denotes its conjugacy class. We will sometimes blur the distinction between $\mathcal{B}$-words (or cyclic $\mathcal{B}$-words) and the elements (or conjugacy classes) that they represent.

Let $R_{\mathcal{B}}$ denote the wedge of $|\mathcal{B}|$ oriented circles with fundamental group identified with $\mathbb{F}$. $R_{\mathcal{B}}$ is an example of a *labeled graph*. More generally, a labeled graph is a connected non-empty finite graph[6] Γ together with a combinatorial[7] map $l : \Gamma \to R_{\mathcal{B}}$ called a *labeling*. We consider two labelings l and l' to be the same if for each edge e, the paths $l|e$ and $l'|e$ are homotopic rel endpoints. Thus, a labeling is equivalent to a choice of $u(e) \in \mathcal{B}^{\pm 1} := \mathcal{B} \sqcup \mathcal{B}^{-1}$ for each oriented edge e of Γ such that $u(e^{-1}) = u(e)^{-1}$ where e^{-1} is the edge opposite to e. A labeling also induces labelings of edge paths in Γ.

If $l : \Gamma \to R_{\mathcal{B}}$ is an immersion and if Γ has no valence 1 vertices then we say that l or Γ is *tight*. A *morphism* of labeled graphs $l_1 : \Gamma_1 \to R_{\mathcal{B}}$ and $l_2 : \Gamma_2 \to R_{\mathcal{B}}$ is a combinatorial map $f : \Gamma_1 \to \Gamma_2$ that preserves

[5]not a proper power
[6]1-dimensional CW-complex
[7]cellular taking open edges homeomorphically to open edges

labels, i.e. $l_1 = l_2 \circ f$. An injective homomorphism $\phi : \mathbb{F}_1 \to \mathbb{F}_2$ induces a cellular map $R_{\mathcal{B}_1} \to R_{\mathcal{B}_2}$ that immerses each edge. A morphism is obtained by subdividing edges of R_{B_1}. If $l : \Gamma \to R_{\mathcal{B}_1}$ is a labeling then $\phi(l) : \phi(\Gamma) \to R_{\mathcal{B}_2}$ is the induced labeled graph

$$\Gamma \xrightarrow{l} R_{\mathcal{B}_1} \to R_{\mathcal{B}_2}$$

Similarly, if $f : \Gamma_1 \to \Gamma_2$ is a morphism then there is an induced morphism $\phi(f) : \phi(\Gamma_1) \to \phi(\Gamma_2)$.

For a labeling $l : \Gamma \to R_{\mathcal{B}}$, $\mathsf{Im}\,\pi_1(l)$ is a well-defined conjugacy class $\mathcal{H}$ of a subgroup of $\mathbb{F}$ and we say that l is a *labeling* for $\mathcal{H}$ or that l *represents* $\mathcal{H}$. There is a 1-1 correspondence between tight labelings of finite graphs and conjugacy classes of f.g. subgroups of $\mathbb{F}$. A labeling $l : \Gamma \to R_{\mathcal{B}}$ of a finite graph can always be *folded* until it is an immersion, see [6]. Valence one vertices can then be iteratively pruned until it is tight. Let $\tau(l) : \tau(\Gamma) \to R_{\mathcal{B}}$ denote the resulting tight labeling. This tight labeling is unique unless Γ is contractible in which case $\tau(\Gamma)$ will consist of a single vertex.

We now consider the problem of extending a labeling $l : C \to R_{\mathcal{B}}$ of an oriented circle C to a surface. A *bounding of l* is a morphism $b : C \to \Gamma(b)$ that is generically 2-to-1 and generically locally of degree 0, i.e. the b-preimage of an open edge consists of two inconsistently oriented open edges in C. We say the corresponding closed edges are *b-paired*. The mapping cylinder $\mathbb{S}$ of b is a surface with boundary C perhaps with some points identified. Let $\mathcal{NV}(b)$ denote the set of natural vertices of $\Gamma(b)$, i.e. the set of vertices of valence other than 2 and let $\mathcal{NE}(b)$ denote the set of natural edges of $\Gamma(b)$, i.e. the closures of components of $\Gamma(b) \setminus \mathcal{NV}(b)$. Set $v(b) = |\mathcal{NV}(b)|$ and $e(b) = |\mathcal{NE}(b)|$. The *geometric genus* of b

$$\mathsf{g\text{-}genus}\ b := \frac{1}{2} \cdot \big(1 - v(b) + e(b)\big)$$

is half the rank of $H_1(\mathbb{S}; \mathbb{Z})$. If l represents a cyclic $\mathcal{B}$-word w then we also say that b is a *bounding* of w (or of $[[w]]$). The *geometric genus* of the conjugacy class ω of an element in $[\mathbb{F}, \mathbb{F}]$ is

$$\mathsf{g\text{-}genus}\ \omega := \min\{\mathsf{g\text{-}genus}\ b \mid b \text{ is a bounding of } \omega\}$$

The link $Lk_{\Gamma(b)}(v)$ of a vertex v of $\Gamma(b)$ is a union of points, one for each oriented edge with initial endpoint v. For each point $\hat{v}$ in the b-preimage of v there is an induced map $Lk_C(\hat{v}) \to Lk_{\Gamma(b)}(v)$. The *Whitehead graph of v* has vertex set $Lk_{\Gamma(b)}(v)$ and an edge connecting the vertices in the image of $Lk_C(\hat{v}) \to Lk_{\Gamma(b)}(v)$ for each $\hat{v} \in b^{-1}(v)$. If the Whitehead graph of v is not connected, then a new bounding with smaller geometric genus can be constructed in the obvious way by "pulling apart v". If b is a bounding of l with minimal geometric genus then all Whitehead graphs are connected, the mapping cylinder $\mathbb{S}$ of b is a surface with boundary C (no extra points are identified), and the genus of $\mathbb{S}$ is $\mathsf{g\text{-}genus}\, b$. This is the motivation for the definition of geometric genus. If C is the concatenation of edge paths $p_1 \cdots p_{4g}$ and if the induced edge paths $b_*(p_j)$ and $b_*(p_{j+2}^{-1})$ coincide for $j \equiv 1$ or $2 \mod 4$, then b is a *standard bounding*.

The next lemma and corollary are classical. It can be proved, for example, using cut-and-paste surface techniques and folding.

Lemma 2.1. *Let $b : C \to \Gamma(b)$ be a bounding for the labeling $l : C \to R_{\mathcal{B}}$ representing the cyclic $\mathcal{B}$-word w.*

1. *Recall that $\tau(l) : \tau(C) \to R_{\mathcal{B}}$ is the labeling obtained by tightening l. There is a bounding denoted $\hat{b} : \tau(C) \to \Gamma(\hat{b})$ for $\tau(l)$ with $\mathsf{g\text{-}genus}\, \hat{b} \leq \mathsf{g\text{-}genus}\, b$.*
2. *There is a labeled graph $l' : C' \to R_{\mathcal{B}}$ representing $[[w]]$ with a standard bounding $b' : C' \to \Gamma(b')$ such that $\mathsf{g\text{-}genus}\, b' = \mathsf{g\text{-}genus}\, b$.* □

Warning 2.2. The labeled graph $\Gamma(\hat{b})$ in Lemma 2.1(1) need not be tight. Even though $\tau(C)$ is tight and therefore $\hat{b}$ is an immersion, it is possible that, after a fold of $\Gamma(\hat{b})$, the induced map from $\tau(C)$ is no longer generically 2-to-1 and therefore not a bounding. Folding at a 4-pronged singularity (see Figure 4) would be an example. Note however that no folding is possible at a valence two vertex of $\Gamma(\hat{b})$.

Here is an example of a bounding for l and a corresponding bounding for $\tau(l)$.

Example 2.3. Suppose $\mathcal{B} = \{u, v, w\}$ and let $l : C \to \mathbb{R}_{\mathcal{B}}$ represent the cyclic word $[uv, wU]$. The labeling l has a standard bounding $b : C \to \Gamma(b)$ where $\Gamma(b)$ is a wedge of two circles, one labeled uv and the other wU. The labeling $\tau(l) : \tau(C) \to R_B$ represents the cyclic word $uvwUVW$ and has the bounding $\tau(b)$ illustrated in Figure 1.

Corollary 2.4. *For $x \in [\mathbb{F}, \mathbb{F}]$, $\mathsf{a\text{-}genus}\, x = \mathsf{g\text{-}genus}\, x$.*

Definition 2.5. For $x \in [\mathbb{F}, \mathbb{F}]$, the *genus of x* denoted $\mathsf{genus}\, x$ is the number $\mathsf{a\text{-}genus}\, x = \mathsf{g\text{-}genus}\, x$. Similarly $\mathsf{genus}\, [[x]] := \mathsf{a\text{-}genus}\, [[x]] = \mathsf{g\text{-}genus}\, [[x]]$.

We record the next easy lemma for later use.

Lemma 2.6. *Let $b : C \to \Gamma(b)$ be a bounding for the labeling $l : C \to R_{\mathcal{B}}$.*

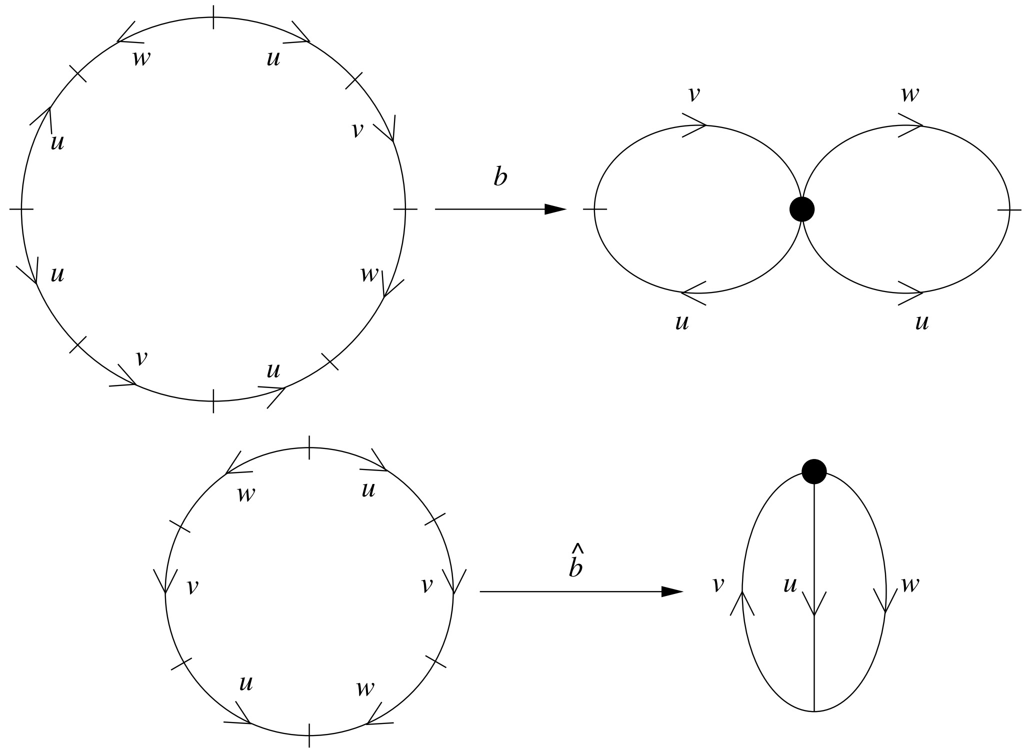

FIGURE 1. A bounding of a labeled graph and the corresponding bounding of its tightening

(1) *Let b' be a new bounding for a new labeling obtained by collapsing an edge of $\Gamma(b)$ and its b-preimage. Then,* $\textsf{g-genus}\, b' \leq \textsf{g-genus}\, b$.
(2) *If l represents a cyclically reduced word then $\Gamma(b)$ has no valence one vertices. In particular,* $v(b) \leq 4 \cdot \textsf{g-genus}\, b - 2$ *and* $e(b) \leq 6 \cdot \textsf{g-genus}\, b - 3$. □

The inequalities in (2) follow from $2 \cdot (\textsf{g-genus}\, b) = 1 - v(b) + e(b)$ and $3v(b) \leq 2e(b)$.

3. Genus 1

Here $\mathcal{B} = \{u, v\}$ and so $\mathbb{F}$ is a free group of rank 2. We use the convention that if w is a $\mathcal{B}$-word then W denotes its inverse.

Proposition 3.1 (Lyndon-Wicks[3])**.** $f'_{\mathbb{F}}(1) \geq 2$. *Specifically, if ψ is the representation given by*

$$u \mapsto uvuvv, v \mapsto UUVU$$

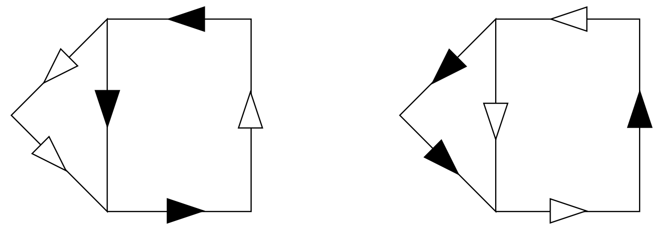

FIGURE 2. The tight labelings of $\mathsf{Im}\,\psi$ and $\mathsf{Im}\,\theta$.

and if θ is given by

$$u \mapsto vuvv, v \mapsto UUVUV$$

then ψ and $i_u \circ \theta$ are inequivalent admissible representations for

$$uvuvvUUVUVVu = [\psi(u), \psi(v)] = i_u([\theta(u), \theta(v)])$$

The proof of Proposition 3.1 will rely on two lemmas.

Lemma 3.2. *$\mathsf{Im}\,\psi$ and $\mathsf{Im}\,\theta$ are not conjugate.*

Proof. The tight labelings representing the conjugacy classes of $\mathsf{Im}\,\psi$ and $\mathsf{Im}\,\theta$ are pictured in Figure 2. Since these labelings are not isomorphic, $\mathsf{Im}\,\psi$ and $\mathsf{Im}\,\theta$ are not conjugate. □

Lemma 3.3. *$\mathsf{Im}\,\psi$ and $\mathsf{Im}\,\theta$ are primitive.*

Proof. If $\phi \in \mathsf{Aut}(\mathbb{F})$ interchanges u and v then $\phi(\mathsf{Im}\,\psi) = \mathsf{Im}\,\theta$. So, it is enough to argue that ψ is primitive. We will show that $\mathsf{Im}\,\psi$ is malnormal in $\mathbb{F}$, i.e. that if $w \in \mathbb{F}$ satisfies $i_w(\mathsf{Im}\,\psi) \cap \mathsf{Im}\,\psi \neq \{1\}$ then $w \in \mathsf{Im}\,\psi$. This clearly implies that $\mathsf{Im}\,\psi$ is primitive. The pullback of two copies of the tight labeling for $\mathsf{Im}\,\psi$ has only one component that is not contractible–that of the "diagonal". From [6], it follows that $\mathsf{Im}\,\psi$ is malnormal in $\mathbb{F}$. □

Proposition 3.1 is proved. □

The homomorphisms ψ and θ in Proposition 3.1 were found by a computer search. The original homomorphisms found by Lyndon and Wicks were ψ' given by

$$u \mapsto uvuvUvuvu, v \mapsto vuvuvUvuvUvuvuv$$

and θ' given by

$$u \mapsto vuvUvuvuvuvuvUvuv, v \mapsto UvuvuvU$$

It is easy to check that $[\psi'(u), \psi'(v)]$ and $[\theta'(u), \theta'(v)]$ are conjugate. They argue that $\mathsf{Im}\,\psi'$ and $\mathsf{Im}\,\theta'$ are primitive and point out that the abelianizations of ψ' and θ' are not in the same $SL_2\mathbb{Z}$-orbit. Hence ψ' and θ' are not equivalent.

4. Higher genus

Here we prove:

Proposition 4.1.

$$f'_{\mathbb{F}}(m+n) \geq f'_{\mathbb{F}}(m) f'_{\mathbb{F}}(n)$$

Definition 4.2. Let $\mathbb{F}_1$ and $\mathbb{F}_2$ be two nonabelian free groups with fixed finite bases $\mathcal{B}_1$ and $\mathcal{B}_2$. For a homomorphism $\phi : \mathbb{F}_1 \to \mathbb{F}_2$, set $m(\phi) = \min\{\mathsf{length}\, \phi(u) \mid u \in \mathcal{B}_1\}$ where length is with respect to $\mathcal{B}_2$. We say that ϕ is an α-map (for some $\alpha > 0$) if

- for all $u \in \mathcal{B}_1$, a subword of $\phi(u)$ of length $\geq \alpha m(\phi)$ appears exactly once as a subword of $\phi(u)$, and
- for $u, v \in \mathcal{B}_1^{\pm 1}$, if $\phi(u)$ and $\phi(v)$ have subwords of length $\geq \alpha m(\phi)$ that are isomorphic preserving orientation, then $u = v$.

The idea of α-maps goes back to Sacerdote [4].

Example 4.3. Say $\mathbb{F}_1 = \mathbb{F}_2 = \langle u, v \rangle$. Let

$$\phi(u) = uvu^2vu^3v \cdots u^n v$$

and

$$\phi(v) = uv^2u^2v^2u^3v^2 \cdots u^n v^2$$

As $n \to \infty$, this is an α-map for $\alpha \to 0$.

While working with an α-map $\phi : \mathbb{F}_1 \to \mathbb{F}_2$ the natural unit of length is $\alpha m(\phi)$. We say that an edge path in a $\mathcal{B}_2$-labeled graph or a $\mathcal{B}_2$-word is *n-long* if it has length at least $n\alpha m(\phi)$. Otherwise it is *n-short*.

Lemma 4.4. *Set $\alpha = 1/4$. For all α-maps $\phi : \mathbb{F}_1 \to \mathbb{F}_2$, the following holds.*

(1) *ϕ is injective.*
(2) *For all $x, x' \in \mathbb{F}_1$, x and x' are $\mathbb{F}_1$-conjugate if and only if $\phi(x)$ and $\phi(x')$ are $\mathbb{F}_2$-conjugate.*
(3) *For all $x \in \mathbb{F}_1$ and subgroups S of $\mathbb{F}_1$, x is conjugate into S if and only if $\phi(x)$ is conjugate into $\phi(S)$.*
(4) *For all f.g. subgroups S and S' of $\mathbb{F}_1$, S is $\mathbb{F}_1$-conjugate to S' if and only if $\phi(S)$ is $\mathbb{F}_2$-conjugate to $\phi(S')$.*
(5) *For all $x \in \mathbb{F}_1$, x is indivisible in $\mathbb{F}_1$ if and only if $\phi(x)$ is indivisible in $\mathbb{F}_2$.*
(6) *For all subgroups S of $\mathbb{F}_1$, $\phi(S)$ is primitive in $\mathbb{F}_2$ if and only if S is primitive in $\mathbb{F}_1$.*

Proof. (1): Here $\alpha < 1/2$ works. Let $x = u_1 \dots u_N$ represent a cyclically reduced cyclic $\mathcal{B}_1$-word. The cyclic word $\phi(x) = \phi(u_1) \cdot \ldots \cdot \phi(u_N)$ is nearly cyclically reduced in that cancellations only occur in a $m(\phi)/2$-neighborhood of the "$\cdot$"'s. Since $\alpha < 1/2$, for each i, not all of $\phi(u_i)$ cancels and $\phi(x)$ is not trivial.

(2): The "$\Longrightarrow$" direction is obvious and holds for any homomorphism $\mathbb{F}_1 \to \mathbb{F}_2$. For the other direction, assume $[[\phi(x)]] = [[\phi(x')]]$. Let $l : C \to R_{\mathcal{B}_1}$ be a labeling representing $x = u_1 \dots u_N$ and let $l' : C' \to R_{\mathcal{B}_1}$ represent $x' = u'_1 \dots u'_{N'}$ as cyclically reduced cyclic $\mathcal{B}_1$-words. The labelings $\phi(l) : \phi(C) \to R_{\mathcal{B}_2}$ and $\phi(l') : \phi(C') \to R_{\mathcal{B}_2}$ represent respectively the $\mathcal{B}_2$-words $\phi(u_1) \cdot \phi(u_2) \cdot \dots \cdot \phi(u_N)$ and $\phi(u'_1) \cdot \phi(u'_2) \cdot \dots \cdot \phi(u'_{N'})$. As in the proof of (1), the labelings $\phi(l)$ and $\phi(l')$ are nearly tight in that, in cyclically reducing $\phi(x)$ and $\phi(x')$, cancellations occur only in an $m(\phi)/4$-neighborhood of the "$\cdot$"s. In particular, there are 2-long subwords p_i of $\phi(u_i)$ and p'_j of $\phi(u'_j)$ that survive the reduction with $\tau(\phi(l))$ and $\tau(\phi(l'))$ representing the same cyclic words $p_1 \dots p_N = p'_1 \dots p'_{N'}$.

Claim: If p_i and p'_j share a 1-long subword p then $p_i = p'_j$.

Before proving the claim, we show that it implies (2). The p_i's and p'_j's are 2-long and so some p_i shares a 1-long subword with some p'_j. By the claim, $p_i = p'_j$. Up to a cyclic permutation, we may assume that $i = j = 1$. Then p_2 and p'_2 share a 1-long subword and $p_2 = p'_2$, etc.

We now prove the claim. We may assume that p is chosen to be maximal, i.e. p is contained in no longer shared subword. We will show that $p_i = p = p'_j$. Set $\phi(u_i) = spt$ and $\phi(u'_j) = s'pt'$. Since p is 1-long, Definition 4.2 gives $u_i = u'_j$, $s = s'$, and $t = t'$. Now, $p_i = s_i p t_i$ (so s_i is the subword of s that survives cancellation). Similarly, $p'_j = s'_j p t'_j$. The claim is that s_i, s'_j, t_i, and t'_j are all trivial. Since p is maximal one of s_i and s'_j, say s_i, is the empty word. If s'_j is not also empty then the terminal letter of s'_j and the terminal letter of s are the same letter b and $\phi(u_{i-1})$ contains the subword bB, contradiction[8] . See Figure 3. That t_i and t'_j are trivial is similar.

(3) is a direct consequence of (2). Indeed, if $\phi(x)$ is conjugate into $\phi(S)$ then, for some $s \in S$, $\phi(x)$ is conjugate to $\phi(s)$. By (2) x is conjugate to s.

(4): Suppose that S and S' are finitely generated subgroups of $\mathbb{F}_1$ such that $\phi(S)$ and $\phi(S')$ are conjugate in $\mathbb{F}_2$. Let $l : \Gamma \to R_{\mathcal{B}_1}$ and $l' : \Gamma' \to R_{\mathcal{B}_2}$ be tight labelings representing S and S' respectively. (2)

[8]Recall the convention that corresponding small and capital letters are mutually inverse.

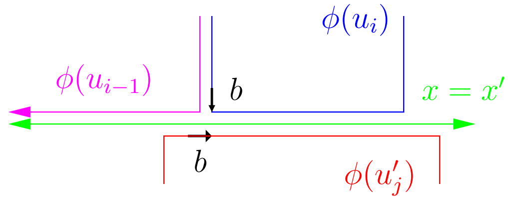

FIGURE 3. Adjacent parallel segment should be viewed as overlapping.

is a special case with $\Gamma = C$ and $\Gamma' = C'$. So, we may assume that S and S' are not cyclic.

Consider a natural edge e of Γ viewed as a labeled edge path representing the word $u_1 \dots u_n$. The edge path $\phi(e)$ is a natural edge of the graph $\phi(\Gamma)$ representing $\phi(u_1)\dots\phi(u_N)$. The edge path $\tau(\phi(e))$ nearly represents a natural edge of $\tau(\phi(\Gamma))$. That is, there are 2-long subwords p_i of $\phi(u_i)$ so that $p_1 \cdots p_N$ is a natural edge of $\tau(\phi(\Gamma))$ agreeing with $\tau(\phi(e))$ except perhaps in 1-short initial and terminal subwords. It follows exactly as in (2) that there is a corresponding natural edge of $\phi(\Gamma')$ representing $\phi(u_1)\cdots\phi(u_N)$ and (4) follows.

(5): The "$\Longleftarrow$" direction is obvious. For the other direction, let $l : C \to R_{\mathcal{B}_1}$ represent the cyclically reduced non-trivial indivisible cyclic word $x = u_1 \dots u_N$ and suppose that $\tau(\phi(l)) : \tau(\phi(C)) \to R_{B_2}$ represents $[[\phi(x)]] = y^n$ with $n > 1$ maximal and y cyclically reduced. Rotation by $2\pi/n$ induces a (label preserving) isomorphism $\rho : \tau(\phi(C)) \to \tau(\phi(C))$. As in (2), $y^n = p_1 \cdots p_N$ where p_i is the 2-long subword of $\phi(u_i)$ that survives cancellation. If we set $p'_i = \rho(p_i)$ then p_i shares a 1-long subword with some p'_j. Exactly as in (2), $p_i = p'_j$. It follows that ρ leaves the set of p_i's invariant and that x is not indivisible, contradiction.

(6) follows directly from (5). □

Example 4.5. We have seen an α-map $\mathbb{F}_1 \to \mathbb{F}_2$ with $\alpha < 1/3$ is injective and also induces an injection $\mathcal{C}(\mathbb{F}_1) \to \mathcal{C}(\mathbb{F}_2)$. Of course, not all homomorphisms have this property. For example, suppose that $\mathcal{B} = \{a, b\}$ and let $\phi(a) = a$, $\phi(b) = baB$, then ϕ is injective and $[[\phi(a)]] = [[\phi(b)]]$, yet $[[a]] \neq [[b]]$.

Lemma 4.6. *Let $x \in \mathbb{F}_1$ have genus g. There is $\alpha > 0$ such that, for all α-maps $\phi : \mathbb{F}_1 \to \mathbb{F}_2$, $\phi(x)$ has genus g.*

Proof. Suppose $x = u_1 \cdots u_M \in \mathbb{F}_1$ is cyclically reduced and has genus g. Represent $u_1 \cdots u_M$ by a tight labeling $l : C \to R_{\mathcal{B}_1}$ (so C has M edges). Choose $\alpha < [4M(16g - 8 + M)]^{-1}$. This reason for this choice will become clear later. Let $\phi : \mathbb{F}_1 \to \mathbb{F}_2$ be an α-map and set

$m := m(\phi)$. Consider the induced labeling $\phi(l) : \phi(C) \to R_{\mathcal{B}_2}$ (so $\phi(C)$ has $|\phi(u_1)| + \cdots + |\phi(u_M)|$ edges). We can identify subwords of $\phi(u_i)$ in $\phi(u_1) \cdots \phi(u_M)$ with certain edge paths in $\phi(C)$. If w_i is a subword of some $\phi(u_i)$ and if u_i equals u_j or U_j then there is a *corresponding* subword w_j of $\phi(u_j)$ or $\phi(U_j)$. More formally, if w_i (respectively w_j) is represented by the edge path $p_i : I \to \phi(C)$ (respectively p_j) then w_i and w_j *correspond* if the edge paths $\phi(l) \circ p_i$ and $\phi(l) \circ p_j$ in $\phi(R_{\mathcal{B}_1})$ are equal.

As in Lemma 4.4, $\phi(l)$ is almost tight and $\tau(\phi(l))$ is obtained by folding $\phi(l)$ in 1-short neighborhoods of at most M of the vertices of $\phi(C)$. Suppose that $\tau(\phi(l))$ represents the cyclically reduced word $v_1 \cdots v_M$ where each v_i is the surviving subword of $\phi(u_i)$ (so $\tau(\phi(C))$ has $|v_1| + \cdots + |v_M|$ edges). Since $\alpha < 1/4$, the length of each v_i is at least $m/2$. In order to obtain a contradiction, assume that $\tau(\phi(l))$ has a bounding $b_{\tau(\phi(l))}$ with geometric genus $g_{\tau(\phi(l))}$ less than g (see Lemma 2.1(1)). Our ultimate goal is to obtain a bounding for x of geometric genus $\leq g_{\tau(\phi(l))}$. By Lemma 2.6(2), $\Gamma(b_{\tau(\phi(l))})$ has no valence 1 vertices, $v(b_{\tau(\phi(l))}) < 4g - 2$, and $e(b_{\tau(\phi(l))}) < 6g - 3$. The natural edges of $\Gamma(b_{\tau(\phi(l))})$ are labeled with $\mathcal{B}_2$-subwords of $v_1 \cdots v_M$ and, as above, we can talk of their lengths. We may also identify the v_i's with edge subpaths of $\phi(C)$ via the labeling $\phi(l)$. The proof of this lemma will be more involved than that of Lemma 4.4 primarily because some of these natural edges may be 1-short and $\Gamma(b_{\tau(\phi(l))})$ need not be tight (see Warning 2.2). The proof consists of three steps.

Step 1. (Find a bounding $b_{\phi(l)}$ of $\phi(l)$ with geometric genus at most $g_{\tau(\phi(l))}$ such that $b_{\phi(l)}$-paired edges correspond.) Consider a point y in a natural edge e of $\Gamma(b_{\tau(\phi(l))})$ whose distance from $\mathcal{NV}(b_{\tau(\phi(l))})$ is at least $4\alpha m$. Since the length of each v_i is more than $m/2$ and $\alpha < 1/8$, the $b_{\tau(\phi(l))}$-image of some v_j meets e in a 2-long maximal subpath p containing y, i.e. if we view v_j as a path in $\Gamma(b_{\tau(\phi(l)})$ then p is the maximal common subpath of v_j and e containing y. Further, the $b_{\tau(\phi(l))}$-image of some V_k, $k \neq j$ shares a maximal 1-long subpath q with p. Arguing exactly as in Lemma 4.4(2), $p = q$ and the maximal common subpaths of v_j and V_k (again viewed as paths in $\Gamma(b_{\tau(\phi(l))})$) in e and containing p (equivalently y) correspond. We conclude that an edge of $\tau(\phi(C))$ whose $b_{\tau(\phi(l))}$-image contains a point outside the $4\alpha m$-neighborhood of $\mathcal{NV}(b_{\tau(\phi(l))})$ corresponds with its $b_{\tau(\phi(l))}$-paired edge. In particular, the number of edges of $\tau(\phi(C))$ not corresponding with their $b_{\tau(\phi(l))}$-paired edge is at most $8\alpha m \cdot v(b_{\tau(\phi(l))}) < 8\alpha m(4g - 2)$.

The difference in the number of edges of $\phi(C)$ and $\tau(\phi(C))$ is at most $2\alpha m M$. Viewing the edges of $\tau(\phi(C))$ as edges of $\phi(C)$, we have a pairing of corresponding edges of $\phi(C)$ except for at most $8\alpha m(4g-2)+2\alpha mM = 2\alpha m(16g-8+M)$ edges. Edges that are paired by this partial pairing correspond. We want a *saturated* partial pairing, i.e. we want the additional property that if an edge is unpaired then all corresponding edges are unpaired. This can be obtained by taking our partial pairing and forgetting pairings of all edges that correspond to an unpaired edge. Since an edge has at most M corresponding edges, we now have a saturated partial pairing of corresponding edges of $\phi(C)$ except for at most $2\alpha mM(16g-8+M) < m/2$ edges. Since $|v_i| \geq m/2$, in each $\phi(u_i)$ there is at least one paired edge. This explains our choice of α. If we collapse unpaired edges we get a bounding b' with $\mathsf{g\text{-}genus}\, b' \leq g_{\tau(\phi(l))}$ by Lemma 2.1(2). Since our partial pairing is saturated, it can be extended to the sought-after bounding $b_{\phi(l)}$ of $\phi(l)$ with $\mathsf{g\text{-}genus}\, b_{\phi(l)} = \mathsf{g\text{-}genus}\, b'$. Here's how.

Recall that $\phi(l)$ represents $\phi(u_1)\dots\phi(u_M)$ and we may view the $\phi(u_i)$'s as edge paths in $\phi(C)$. Suppose that p is a first maximal unpaired subpath of some $\phi(u_i)$. Since $\phi(u_i)$ contains a paired edge, an edge w of p shares an endpoint with an edge q of $\phi(u_i)$ on which our partial pairing is defined. Our partial pairing is defined on all edges of $\phi(C)$ corresponding to q and determines a pairing on edge paths corresponding to w as follows. If q_1 and q_2 are paired edges corresponding to q and if w_k corresponds to w and shares an endpoint with q_k, $k=1,2$, then pair w_1 with w_2. In this way, we extend our partial pairing. The extended partial pairing is still saturated and has fewer unpaired edges. Further, if we now collapse unpaired edges then we get a pairing b'' such that $\Gamma(b')$ is obtained from $\Gamma(b'')$ by collapsing disjoint partial natural edges. In particular, $\mathsf{g\text{-}genus}\, b'' = \mathsf{g\text{-}genus}\, b'$. Continue until there are no unpaired edges. This completes Step 1.

Step 2. (Find a bounding of $\phi(l)$ of geometric genus less than g that pairs $\phi(u_i)$'s with $\phi(U_j)$'s.) We start with $b_{\phi(l)}$ found in Step 1 and may assume that Whitehead graphs of vertices in $\Gamma(b_{\phi(l)})$ are connected (see Section 2). If, for each natural vertex v of $\Gamma(b_{\phi(l)})$, $b^{-1}_{\phi(l)}(v)$ consists of initial vertices of $\phi(u_i)$'s then $b_{\phi(l)}$ would be the desired bounding. A natural vertex v not having this property is a $2k$-*pronged singularity* which we now describe. Let $\mathcal{U}$ be the set of u_i's such that $b_{\phi(l)}(\phi(u_i))$ contains v (necessarily as an interior vertex). Restricting $b_{\phi(l)}$ gives a partial pairing on $\cup\{\phi(u) \mid u \in \mathcal{U}\}$. Let $\widetilde{N(v)}$ be the domain of this partial pairing, i.e. the subset of points y in $\cup\{\phi(u) \mid u \in \mathcal{U}\}$ with $|b^{-1}_{\phi(l)}(b_{\phi(l)}(y))| > 1$. Since the Whitehead graph of v is connected, $N(v) := b_{\phi(l)}(\widetilde{N(v)})$ is a closed neighborhood of v that is homeomorphic

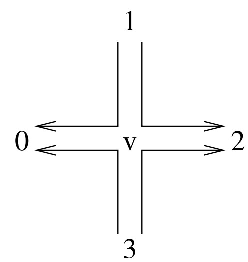

FIGURE 4. Four corresponding paths fitting together to form a 4-pronged singularity at vertex v with an induced cyclic order.

to a cone over an even number, say $2k$, of points with cone point v. This is a $2k$-*pronged singularity.*, see Figure 4.

Since the Whitehead graph of v is connected, the pairing imparts a cyclic order to the natural edges of $N(v)$. There are two choices for this cyclic order, one the inverse of the other. By choosing a 0^{th} edge, we may talk of even edges and odd edges. The outgoing even edges are identically labeled, say by the $\mathcal{B}_2$-word w_0, as are the outgoing odd edges, say by w_1. We may obtain a new bounding of $\phi(l)$ by collapsing odd edges of $N(v)$, relabeling outgoing even edges by W_1w_0, and pulling apart any vertices with disconnected Whitehead graph. The graph of the new bounding either has fewer natural edges or the same number of natural edges and fewer singularities. We then repeat with the new bounding and continue until there are no singularities.

Step 3. (Conclusion) The bounding of $\phi(l)$ found in Step 2 pulls back to a bounding of l with the same geometric genus which is less than g. This is the desired contradiction. □

Corollary 4.7. *For a fixed $x \in \mathbb{F}_1$ there is $\alpha > 0$ such that, for any α-map $\phi : \mathbb{F}_1 \to \mathbb{F}_2$,*

- $\mathsf{genus}\,\phi(x) = \mathsf{genus}\,x$,
- $\mathsf{num}'\,\phi(x) \geq \mathsf{num}'\,x$, *and*
- $\mathsf{num}\,\phi(x) \geq \mathsf{num}\,x$.

In particular, $f'_{\mathbb{F}}(g)$ and $f_{\mathbb{F}}(g)$ do not depend on $\mathbb{F}$.

Proof. The first item is a restatement of Lemma 4.6. For the second item, choose $\alpha < 1/4$ and such that $\mathsf{genus}\,\phi(x) = \mathsf{genus}\,x$. If ψ is an admissible representation for x then $\phi \circ \psi$ is an admissible representation for $\phi(x)$ by Lemma 4.4(6) and the injectivity of α-maps. By Lemma 4.4(4), the map induced by ϕ on conjugacy classes of finite subgroups is injective. Hence, $\mathsf{num}'\,\phi(x) \geq \mathsf{num}'\,x$.

For the third item and the same choice of α, let ψ and θ be representations for x. Suppose $\phi \circ \psi \sim \phi \circ \theta$. So, there is a sequence

$$\psi'_0, \psi'_1, \cdots, \psi'_k$$

of boundings for $\phi(x)$ where $\psi'_0 = \phi \circ \psi$, $\psi'_k = \phi \circ \theta$, and ψ'_{i+1} is obtained from ψ'_i either by post-composition with $i_{z'}$ where $z' \in \mathbb{F}_2$ centralizes $\phi(x)$ or by a fractional Dehn twist.

Suppose by induction that $\psi'_i = \phi \circ \psi_i$ for some $\psi_i \sim \psi$. Suppose also that $\psi'_{i+1} = i_{z'} \circ \psi'_i$ where z' centralizes $\phi(x)$, i.e. z' and $\phi(x)$ are powers of some indivisible $\hat{z}' \in \mathbb{F}_2$. Since $\alpha < 1/3$, Lemma 4.4(5) can be applied to show that $\hat{z}' = \phi(\hat{z})$ for some indivisible $\hat{z} \in \mathbb{F}_1$ centralizing x. Thus $\psi'_{i+1} = \phi \circ i_z \circ \psi_i = \phi \circ \psi_{i+1}$ for some $z \in \mathbb{F}_1$ centralizing x and some $\psi_{i+1} \sim \psi$.

The case where ψ'_{i+1} is obtained from ψ'_i by a fractional Dehn twist is similar and left to the reader. We conclude that $\psi \sim \theta$. Hence $\mathsf{num}\,\phi(x) \geq \mathsf{num}\,x$.

For the final statement, let $x \in \mathbb{F}_1$ also satisfy $f_{\mathbb{F}}(g) = \mathsf{num}\,x$ then

$$f_{\mathbb{F}_1}(g) = \mathsf{num}\,x \leq \mathsf{num}\,\phi(x) \leq f_{\mathbb{F}_2}(g)$$

Since $\mathbb{F}_1$ and $\mathbb{F}_2$ were arbitrary, $f_{\mathbb{F}_1}(g) = f_{\mathbb{F}_2}(g)$. The case of $f'_{\mathbb{F}}$ is similar. □

Proof of Proposition 4.1. Let $x \in \mathbb{F}$ and $y \in \mathbb{F}$, cyclically reduced, have genera m and n realizing $f'_{\mathbb{F}}(m)$ and $f'_{\mathbb{F}}(n)$. Consider $z = xy \in \mathbb{F} * \mathbb{F}$. Thus $\mathsf{genus}\,z = m + n$ and $\mathsf{num}'\,z \geq \mathsf{num}'\,x \cdot \mathsf{num}'\,y = f'_{\mathbb{F}}(m) \cdot f'_{\mathbb{F}}(n)$. For an α-map $\phi : \mathbb{F} * \mathbb{F} \to \mathbb{F}$ with small α we have

$$\mathsf{num}'\,\phi(z) \geq \mathsf{num}'\,z \geq f'_{\mathbb{F}}(m) f'_{\mathbb{F}}(n)$$

and thus $f'_{\mathbb{F}}(m + n) \geq f'_{\mathbb{F}}(m) f'_{\mathbb{F}}(n)$. □

Remark 4.8. We discovered a new limit group quotient that does not factor through any of the obvious quotients. For example, take G to be the union of 4 genus 2 surfaces with one boundary component along their boundaries. Take L to be the wedge of two genus two surfaces. Map $G \to L$ by sending the common boundary to the product of the two waist curves, and sending each genus two membrane to the "boundary connected sum" of two halves (there are 4 possible combinations – use all 4).

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