# Homework on $Out(F_n)$

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Problems labeled with an asterisk are more difficult/technical and constitute the take-home final.

## 1 Folding and applications

$H$ is a finitely generated subgroup of $F_n$.

1. Find a basis of the subgroup
$$H = \langle b\overline{a}baaba, ab\overline{a}baba, ab\overline{a}b\overline{a}\overline{b}a\overline{b}\rangle < \langle a, b\rangle$$
2. Given $w \in F_n$ give an algorithm to decide whether $w \in H$. E.g. show $a \notin H$ of #1.
3. Given $w \in F_n$ give an algorithm to decide whether $w$ is conjugate into $H$.
4. Can you tell if $H$ is normal in $F_n$?
5. Can you tell if $H$ has finite index in $F_n$?
6. Suppose $H$ is a finitely generated normal subgroup of $F_n$. Show that either $H$ has finite index in $F_n$ or $H = \{1\}$.
7. Given a homomorphism $h : F_n \to F_m$, can you tell if $h$ is injective, surjective, bijective? Answer: Injective iff there are no folds of the second kind. Surjective iff the last map is a homeomorphism. In particular, show that $F_n$ is *hopfian*, i.e. every epimorphism $F_n \to F_n$ is an automorphism.

8. Let $h : \langle a, b\rangle \to \langle a, b\rangle$ be given by $h(a) = abbab$, $h(b) = bababbab$. Show that $h$ is an automorphism and compute $h^{-1}$. (You can do this by messing about. But try to do it algorithmically, that is, decompose $h$ into a product of Nielsen generators and then compose the inverses in opposite order. The point is that this can be programmed on a computer.)

9. Show that for every homomorphism $h : F_n \to F_m$ there is a free factorization $F_n = A * B$ such that $h$ kills $A$ and is injective on $B$.

10. Show that for every finitely generated $H \subset F_n$ there is a subgroup $H' \subset F_n$ such that $H \subset H'$, $H$ is a free factor in $H'$, and $H'$ has finite index in $F_n$. This is called Marshall Hall's theorem. You can find $H'$ algorithmically. Do it for $H$ in the example from #1. Hint: Add some edges to $G$ to turn an immersion $G \to Y$ into a covering map.

11. Can you always compute the normalizer

$$N(H) = \{\gamma \in F_n \mid \gamma H \gamma^{-1} = H\}?$$

What can you say about the index $[N(H) : H]$? (Answer: it is always finite and bounded by the number of vertices in the graph representing $H$. Recall that $N(H)/H$ is the deck group.) E.g. show that $N(H) = H$ for $H$ as in #1.

12. If $T$ and $T'$ are two maximal trees, show that there is a sequence $T = T_0, T_1, \cdots, T_k = T'$ of maximal trees such that any two consecutive trees differ in only one edge, as in the lecture.

13.* This is a bit more ambitious. Consider the simplicial complex whose vertices are non-closed edges of $G$, and a collection of edges spans a simplex if their union is a forest. Draw some examples. Can you make a conjecture about the homotopy type of the complex?

14.* Read the wonderful paper *Topology of finite graphs* by John Stallings (Inventiones 71 (1983) 551-565.)

## 2 Outer space

15. Consider a graph with two vertices and four edges, all joining the two vertices. Once a marking is provided, this graph defines a simplex with missing faces in Outer space $\mathcal{CV}_3$. How many faces are missing? How many simplices-with-missing-faces in $\mathcal{CV}_3$ contain this simplex?

16. In $\mathcal{CV}_2$, sketch an orbit of the automorphism $a \mapsto a$, $b \mapsto ab$, and also of $a \mapsto b$, $b \mapsto ab$.

17. Prove that $\mathbb{Z}^m$ cannot act freely and properly discontinuously on a contractible complex of dimension $< m$. Deduce that no subgroup of $Out(F_n)$ is isomorphic to $\mathbb{Z}^{2n-2}$ $(n \geq 2)$ (recall that the dimension of the spine is $2n-3$). Find a subgroup isomorphic to $\mathbb{Z}^{2n-3}$.

18. The smallest dimension of a $\Sigma(\Gamma)$ is $n-1$. (Recall that $\Sigma(\Gamma)$ is the space of normalized nondegenerate metrics on the graph $\Gamma$ of rank $n$.)

19. The largest dimension of a $\Sigma(\Gamma)$ is $3n-4$.

20. $Out(F_n)$ has finitely many conjugacy classes of finite subgroups. (Use Nielsen realization.)

21. Find a nontrivial element of finite order in the kernel of $Out(F_n) \to GL_n(\mathbb{Z}/2)$. Show that every such element has order 2 and that therefore every finite subgroup of the kernel is abelian (in fact, a direct sum of $\mathbb{Z}/2$'s). Can you find the largest such subgroup?

22. Can you find estimates on the size of the largest finite subgroup of $Out(F_n)$? For example, the stabilizer of a rose has order $2^n n!$. Can you find a larger finite group? What about $n = 2$ and 3? For the answer see Wang-Zimmermann: *The maximum order of finite groups of outer automorphisms of free groups.* Math Z., 216, 1994, 83-87.

## 3 Lipschitz metric and train tracks

23. For $\epsilon > 0$ find examples of graphs $\Gamma, \Gamma'$ such that $d(\Gamma, \Gamma') < \epsilon$ and $d(\Gamma', \Gamma) > 1/\epsilon$.

24. Let $R \in \mathcal{CV}_2$ be the rose with identity marking and edges of length 1/2. Let $f$ be given by $a \mapsto a$, $b \mapsto ab$. Show that $d(R, f^k(R)) \sim \log k$. (this means $1/C \log k < d(R, f^k(R)) < C \log k$ for some fixed $C > 1$ and all large $k$).

25. For the same $R$ and $f$ given by $a \mapsto b$, $b \mapsto ab$, show that $d(R, f^k(R))$ is bounded above and below by a linear function of $k$. In fact, if $R$ is replaced by a suitable graph in the same simplex, $k \mapsto d(R, f^k(R))$ is a linear function on the nose. Hint: use the train track metric. By connecting consecutive graphs in the orbit with a folding path one gets an *axis* of $f$.

26. For $f$ in #25, show that axes for $f$ and for $f^{-1}$ are distinct lines.

27. For $f$ in #24 find a sequence $\Gamma_k$ with $d(\Gamma_k, f(\Gamma_k)) \to 0$. Prove that for any such sequence for large $k$ there is going to be a proper invariant subgraph (up to homotopy).

28. Show that in any graph with a train track structure with at least two gates at every vertex, there is a legal loop that is either embedded, or it forms a "figure 8" crossing each edge once, or it forms a "dumbbell", crossing edges in the two loops once and edges in the connecting arc twice. See Figure 1.

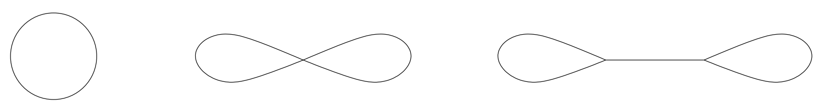

Figure 1: Possible forms of candidates. Train track structure is suggested by the pictures.

We say that an immersed loop in a graph $\Gamma$ (without any train track structure) is a *candidate* if it has a form as in Exercise #28.

29. Let $R_3$ be the rose in $\mathcal{X}_3$ with all edges of length $\frac{1}{3}$ and with inverse marking given by $a, b, c$, and let $\Gamma$ be another such rose but with inverse marking given by $abA, bacB, a$. Find all candidates in each that are witnesses for the distance to the other.

30. Consider the automorphism $\Phi$ of $\mathbb{F}_4 = \langle a, b, c, d\rangle$ given by $a \to b \to c \to d \to ADCB$ (capital letters are inverses of the lowercase letters).

    (a) Let $R$ be the rose with the identity marking (so the edges correspond to $a, b, c, d$) and with all lengths $\frac{1}{4}$. Compute $d(R, R\Phi)$.

    (b) Find the graph $\Gamma$ in the same simplex as $R$ (i.e. the same marking, but edge lengths can be arbitrary) so that $d(\Gamma, \Gamma\Phi)$ is minimal.

    (c) Can you find a graph $\Gamma'$ in a small neighborhood of $\Gamma$ so that $d(\Gamma', \Gamma'\Phi) < d(\Gamma, \Gamma\Phi)$?