Claim: \( H_i(\Delta^n,\partial\Delta^n) = \{ \mathbb{Z}, i = n \} \) generates \( H_n \).

Induction on \( n \). \( n = 0 \) is trivial.

\[ \wedge \in \Delta^n, \text{union of all faces except "bottom".} \]

\[ (\Delta^n,\partial\Delta^n,\wedge) \text{ LES homology} \]

\[ H_i(\Delta^n,\wedge) \rightarrow H_i(\Delta^n,\partial\Delta^n) \rightarrow H_{i-1}(\partial\Delta^n,\wedge) \rightarrow H_{i-1}(\Delta^n,\partial\Delta^n) \]

\[ X = A \cup B \text{ CW complexes} \]

\[ H_i(B,A \cap B) \xrightarrow{\text{exc}} H_i(X,A) \]
2. **Local homology groups**

\( X \ni x, \exists x \in X \text{ closed} \)

\[ H_n(X, X - \{x\}) \] local homology groups at \( x \).

\[ U \ni x \]

\[ H_n(U, U - \{x\}) \cong H_n(X, X - \{x\}) \]

If \( (X, x) \) is homeomorphic to \( (Y, y) \)

\[ \Rightarrow H_n(X, X - x) \cong H_n(Y, Y - y) \]

If \( x \) has a cone nbhd \( cL \)

then \( H_n(cL, cL - \{x\}) \cong H_n(cL, cL - \{x\}) \)

\[ = H_n(cL) \]

**Ex:** \( \mathbb{R}^n \) every \( x \) has a cone nbhd with \( L = S^n \)

\[ H_i(\mathbb{R}^n, \mathbb{R}^n - x) = \begin{cases} 0, & i \neq n \\ \mathbb{Z}, & i = n \end{cases} \]
Cor. \( U \subseteq \mathbb{R}^n, V \subseteq \mathbb{R}^m \) open, \( \neq \emptyset \), homeomorphic \( \Rightarrow n = m \).

\exists \; Y \ni H_1(Y, Y - \frac{\varepsilon}{3} I) \cong \tilde{H}_0(3 \text{ pts}) = \mathbb{Z} \otimes \mathbb{Z}

Cor. \( \bigcirc \infty \) not homeomorphic.

\( \exists I \) if \( (X_\alpha, x_\alpha) \) NDR, connected.

Then \( \bigcirc \oplus \tilde{H}_n(X_\alpha) \cong \tilde{H}_n(V X_\alpha)\)

Pf.

\[ H_n \left( \bigcup X_\alpha, \bigcup \{ x_\alpha \} \right) \cong \tilde{H}_n(V X_\alpha) \]

\[ \{I\} \]

\( \bigcirc \tilde{H}_n(X_\alpha) \)

\( \bigcirc \tilde{H}_n(x_\alpha) \)
4. **Naturality**

\[ f : (X,A) \to (Y,B) \]

\[-\to H_n(A) \to H_n(X) \to H_n(X,A) \overset{f_*}{\to} H_n(A) \to \]

\[-\to H_n(B) \to H_n(Y) \to H_n(Y,B) \overset{f_*}{\to} H_n(B) \to \]

all squares commute.

5. **Attaching a cell.**

\[ Y = X \cup e^n. \]

\[(Y,X) \text{ in NDR} \]

\[ H_{i+m}(Y,X) \to H_i(X) \to H_i(Y) \to H_i(Y,X) \]

If \( i \neq n-m \), \( H_0(X) \xrightarrow{f} H_i(Y) \)
The image of $H_n(Y, X) \to H_{n-1}(X)$ is generated by $f_{*}(z)$, $z \in H_{n-1}(S)$.

**Case 1** $f_{*}(z) = 0$

$$0 \to H_{n}(X) \xrightarrow{f_{*}} H_{n}(Y) \xrightarrow{\partial} H_{n}(Y, X) \to H_{n-1}(X)$$

$$0 \to A \to B \to \mathbb{Z} \to 0$$ splits

$b = A \oplus \mathbb{Z}$

$H_{n}(Y) \cong H_{n}(X) \oplus \mathbb{Z}$, $H_{n-1}(Y) \cong H_{n-1}(X)$

**Case 2** $f_{*}(z)$ has order $k < \infty$

$H_{n}(Y) \cong H_{n}(X) \oplus \mathbb{Z}$

$H_{n-1}(Y) \cong H_{n}(X) \oplus \mathbb{Z}/k\mathbb{Z}$
(code 3) $f_x(z)$ has infinite order.

$0 \to H_n(x) \to H_n(y) \to H_n(Y,x) \to H_{n-1}(y) \to \cdots \to H_0(y) \to 0$

$H_n(x) \cong H_n(x)/Z$

Cor. $CP^n = \mathbb{C}^n \vee e^{2\pi i} \vee \cdots \vee e^{2\pi i n}$

$H_1(CP^n) = \bigoplus \mathbb{Z}$, $i = 0, 2, \ldots, 2n$

$H_1(CP^n) = \{0\}$, otherwise.

6 Euler characteristic

$X$ finite CW complex

$X(X) = \sum_{n=0}^{\dim X} (-1)^n \cdot \#(\text{cells of dim } n)$

$s^n = e^{2\pi i} e^n \quad X(S^n) = \{0\}$, $n$ even

$X(S^n) = \{0\}$, $n$ odd
\[ X(X) = \sum_{i=0}^{\infty} (-1)^i \text{rank} H_i(X) \]

**Pf.** Induction on \# cells.

When adding an \( n \)-cell, RHS increases by \((-1)^n\).