Hilbert Spaces

\( \mathcal{H} \) denotes a Hilbert space.

1. Let \( A, B \subset \mathcal{H} \) be two closed subspaces that are orthogonal to each other. Show that \( A + B \) is a closed subspace. (As Hanlin pointed out correctly, this is false without assuming orthogonality.)

2. Let \( x_n \in \mathcal{H} \) be a sequence and assume that for every \( y \in \mathcal{H} \) the sequence \( \langle x_n, y \rangle \) of inner products converges to some \( z_y \in \mathbb{C} \). Show that there exists \( x \in \mathcal{H} \) such that \( \langle x_n, y \rangle \to \langle x, y \rangle \) for every \( y \in \mathcal{H} \).

3. Let \( f : M \to \mathbb{C} \) be a bounded linear functional on a linear subspace \( M \subset \mathcal{H} \).
   (i) \( f \) has a unique extension to a bounded linear functional \( F : \mathcal{H} \to \mathbb{C} \) of the same norm as \( f \).
   (ii) \( F|M^\perp = 0 \).

4. Assuming \( \dim \mathcal{H} = \infty \) construct a closed subset of \( \mathcal{H} \) that does not contain an element of minimal norm.

5. Let \( \alpha : [0, 1]^2 \to \mathbb{C} \) be continuous. Show that
   \[
   A(f)(x) = \int_0^1 \alpha(x, y)f(y)dy
   \]
   defines a bounded linear operator
   \[
   L^2([0, 1]) \to C([0, 1])
   \]

6. Show that the Parallelogram Law fails in \( L^1([0, 1]) \).

7. Let \( e_1, e_2, \ldots \) be an orthonormal basis of \( \mathcal{H} \).
   (i) Show that 0 is the weak limit of the sequence \( e_1, e_2, \ldots \).
   (ii) \( B = \{ x \in \mathcal{H} \mid ||x|| \leq 1 \} \) is closed in the weak topology.
   (iii) \( S = \{ x \in \mathcal{H} \mid ||x|| = 1 \} \) is dense in \( B \).

8. Let \( P : \mathcal{H} \to \mathcal{H} \) be a self-adjoint linear operator, meaning that \( \langle Px, y \rangle = \langle x, Py \rangle \) for all \( x, y \in \mathcal{H} \). Also assume that \( P^2 = P \). Show that \( P \) is the orthogonal projection to a closed subspace. Note that we are not assuming that \( P \) is bounded.
9. The set of $T \in L(H, H)$ which are isometries is a closed set in the strong operator topology but not necessarily in the weak operator topology.

10. Let $T \in L(H, H)$.

   (i) There is a unique linear $T^* : H \to H$ such that
   \[ \langle Tx, y \rangle = \langle x, T^* y \rangle \]
   for all $x, y \in H$ (Hint: Riesz)

   (ii) $||T^*|| = ||T||$ and $T^{**} = T$.

   (iii) $Im(T)^\perp = Ker(T^*)$ and $Ker(T)^\perp = Im(T^*)$.

   (iv) For $H = \mathbb{C}^n$, identifying $T$ with a complex $n \times n$ matrix in the usual way, $T^*$ is the conjugate-transpose of $T$.

11. Let $H = \ell^2(\mathbb{N})$ and define $T_n : H \to H$ by $T_n(e_i) = e_{n+i}$ for all $i \in \mathbb{N}$. Let $T_n^* : H \to H$ be the adjoint of $T_n$.

   (i) Show that $T_n^*(e_i) = e_{i-n}$ if $i > n$ and $0$ if $i \leq n$.

   (ii) The sequence $T_n$ does not converge in the strong operator topology, and $T_n \to 0$ in the weak operator topology.

   (iii) $T_n^* \to 0$ in the strong operator topology.

Information: Adjoint is continuous in the weak operator topology, but not in the strong operator topology unless $\dim H < \infty$.  

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