Banach-Steinhaus, Riesz, Weak topologies

1. Let $X$ be a compact Hausdorff space and assume that $C(X)$ is equipped with a norm $\| \cdot \|$ that makes it a Banach space. Also suppose that for every $x \in X$ the functional $\lambda_x : C(X) \to \mathbb{R}$ defined by
   $$\lambda_x(f) = f(x)$$
is bounded. Prove that there are $A, B > 0$ such that
   $$A \|f\|_\infty \leq \|f\| \leq B \|f\|_\infty$$
for all $f \in C(X)$ where $\|f\|_\infty = \sup_{x \in X} |f(x)|$ is the standard sup norm.

2. Let $f_n \in C([0,1])$ for $n = 1, 2, \ldots$. Show that the following two statements are equivalent:
   (a) for every $\lambda \in C([0,1])^*$ we have $\lambda(f_n) \to 0$ as $n \to \infty$,
   (b) $f_n(x) \to 0$ for every $x \in [0,1]$ and $\sup \|f_n\|_\infty < \infty$.

3. Let $X$ be compact Hausdorff, $A \subset C(X)$ a linear subspace that contains constant functions. Let $S \subseteq X$ be a compact subset. Suppose that for every $f \in A$
   $$\sup\{|f(x)| \mid x \in X\} = \sup\{|f(x)| \mid x \in S\}$$
Show that for every $x \in X$ there is a positive measure $\mu_x$ supported in $S$ such that
   $$f(x) = \int_S f \, d\mu_x$$
for every $f \in A$.

4. Suppose $V$ is a Banach space and $x, x_n \in V$, $n = 1, 2, \ldots$. Assume $x_n \overset{w}{\to} x$. Prove that there is a constant $C \geq 0$ such that $\|x_n\| \leq C$ for all $n$ and that
   $$\|x\| \leq \liminf_{n \to \infty} \|x_n\|$$

5. This problem illustrates the need to use nets when discussing weak topologies. Let
   $$S = \{\sqrt{n}e_n \mid n = 1, 2, 3, \ldots\} \subset \ell^2$$
where $e_n \in \ell^2$ is the element with $n^{th}$ coordinate 1 and other coordinates 0.
(a) Show that there is no sequence in $S$ that converges to 0 in the weak topology. Hint: #4.

(b) Show that 0 is in the weak closure of $S$ (by the “weak closure” I mean closure in the weak topology). That is, show that every weak neighborhood of 0 contains (infinitely many) elements of $S$.

A primer on nets. A directed set is a partially ordered set $(I, \leq)$ such that for all $i, j \in I$ there is $k \in I$ with $i \leq k, j \leq k$.

Examples: $I = \mathbb{N}$ with the usual ordering. The set $I$ of open sets in a topological space $\Omega$ containing a given point $x$, and ordering $U \leq V$ when $V \subseteq U$.

Let $\Omega$ be a topological space. A net in $X$ is a function $\phi : I \to \Omega$ for a directed set $I$. Instead of $\phi(i)$ we write $x_i$. We say that the net $(x_i)$ converges to $x \in \Omega$ if for every neighborhood $U$ of $x$ there is some $i \in I$ such that $i \leq j$ implies $x_j \in U$. If $Z \subset \Omega$ is closed and $x_i \in Z$ then the limit (if it exists) of the net $(x_i)$ belongs to $Z$.

6. In the setting of #5 show that there is a net in $S$ converging to 0. More generally, show that in any topological space $\Omega$ if $x$ is in the closure of a set $S$ then there is a net in $S$ converging to $x$. Hint: Use a system of neighborhoods of $x$ for the index set.