Jordan measure

1. Let $m' : \mathcal{E} \to [0, \infty)$ be a function defined on the elementary sets in $\mathbb{R}^d$ satisfying:
   
   (i) (finite additivity) $m'(A \sqcup B) = m'(A) + m'(B)$ for disjoint $A, B \in \mathcal{E}$,
   (ii) (translation invariance) $m'(A + x) = m'(A)$ for $A \in \mathcal{E}, x \in \mathbb{R}^d$,
   (iii) (normalization) $m'([0, 1]^d) = 1$.

   Show that $m' = m$ is the standard measure on elementary sets. How do you modify the proof if normalization is stated as $m'([0, 1]^d) = 1$?

   Hint: First show that $m' = m$ on half-open $\frac{1}{N_1} \times \frac{1}{N_2} \times \ldots \times \frac{1}{N_d}$-boxes.

2. Let $m'$ be a function defined on Jordan measurable sets in $\mathbb{R}^d$ with values in $[0, \infty)$ satisfying finite additivity, translation invariance and normalization. Then $m' = m$ agrees with the Jordan measure.

3. Let $f : [a, b] \to \mathbb{R}^+ = [0, \infty)$ be a continuous function. Show that the set

   \[ \{(x, y) \in \mathbb{R}^2 \mid x \in [a, b], 0 \leq y \leq f(x)\} \subset \mathbb{R}^2 \]

   is Jordan measurable.

   Hint: $f$ is uniformly continuous.

4. Show that any
   
   - round disk
   - triangle
   - convex polygon

   in $\mathbb{R}^2$ is Jordan measurable. Can you generalize to any polygon?

5. Let $A \subset \mathbb{R}^d$ such that for every $\epsilon > 0$ there are Jordan measurable sets $B, C$ with $B \subset A \subset C$ such that $m(C) - m(B) < \epsilon$. Show that $A$ is Jordan measurable. This corresponds more closely to the ancient Greeks’ method of inscribing and circumscribing polygons to estimate area.

6. (This one is more rewarding – read “harder” – than the others.) Let $L : \mathbb{R}^d \to \mathbb{R}^d$ be a linear map. Show that if $E \subset \mathbb{R}^d$ is Jordan measurable so is $L(E)$ and

   \[ m(L(E)) = | \det L | \ m(E) \]
Hint: If det $L = 0$ $L(E)$ is contained in a hyperplane. Otherwise reduce to the case when $L$ is an elementary matrix and $E$ is a box. It is useful to consider the function $m'(E) = \frac{m(L(E))}{|\det L|}$ and use Problems 1-3. The picture below suggests how to compute $m(L(E))$ when $E = [0, 1]^2$ and $L = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

7. Show that $[0, 1] \cap \mathbb{Q} \subset \mathbb{R}$ and $[0, 1]^2 \cap \mathbb{Q}^2 \subset \mathbb{R}^2$ are not Jordan measurable.

8. (Another more rewarding problem.) Construct a bounded open set in $\mathbb{R}$ that is not Jordan measurable.

9. (Carathéodory) Let $A \subset \mathbb{R}^d$ be a bounded set and $E \subset \mathbb{R}^d$ an elementary set. Show that

$$m^{*,J}(A) = m^{*,J}(A \cap E) + m^{*,J}(A \setminus E)$$

where $m^{*,J}$ denotes outer Jordan measure.