Question 1 (Artin 3.3.2). Let $W \subset \mathbb{R}^4$ be the space of solutions of the system of linear equations $AX = 0$, where $A = \begin{bmatrix} 2 & 1 & 2 & 3 \\ 1 & 1 & 3 & 0 \end{bmatrix}$. Find a basis for $W$.

Solution. Reducing the rows, we get the following system:

\[
\begin{aligned}
& x - z + 3w = 0 \\
& y + 4z - 3w = 0.
\end{aligned}
\]

Solving each equation for $x$ and $y$ respectively we get the following general expression on the vectors in $W$:

\[
(x, y, z, w)^t = (z - 3w, -4z + 3w, z, w)^t = z(1, -4, 1, 0)^t + w(-3, 3, 0, 1)^t,
\]

so $\langle (1, -4, 1, 0)^t, (-3, 3, 0, 1)^t \rangle$ is a basis for $W$. //

Question 2 (Artin 3.3.8). Prove that a set $(v_1, \ldots, v_n)$ of vectors in $\mathbb{F}^n$ is a basis if and only if the matrix obtained by assembling the coordinate vectors of $v_i$ is invertible.

Proof. Let $A$ be the matrix obtained by assembling the coordinate vectors of $v_i$. Then $(v_1, \ldots, v_n)$ forms a basis if and only if the row space of $A^t$ is the whole space $\mathbb{F}^n$, which is equivalent to saying the system $A^tX = 0$ has unique solution $X = 0$, if and only if $A^t$ is invertible, if and only if $A$ is invertible. \qed

Question 3 (Artin 3.4.1). (a) Prove that the set $B = ((1,2,0)^t, (2,1,2)^t, (3,1,1)^t)$ is a basis of $\mathbb{R}^3$.

(b) Find the coordinate vector of the vector $v = (1,2,3)^t$ with respect to this basis.

(c) Let $B' = ((0,1,0)^t, (1,0,1)^t, (2,1,0)^t)$. Determine the basechange matrix $P$ from $B$ to $B'$.

Proof. (a) Let $P_1 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \\ 0 & 2 & 1 \end{bmatrix}$. Reducing $P_1^t$, we get $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -\frac{2}{5} \\ 0 & 0 & 1 \end{bmatrix}$, having no zero rows.

This implies that the row space of $P_1^t$, which is $\langle B \rangle$, is the whole space $\mathbb{R}^3$. This concludes $B$ is a basis.

(b) Note the matrix $P_1$ above is the basischange matrix from $B$ to the standard one. Because what we want is the other way around, we compute $P_1^{-1} = \frac{1}{7} \begin{bmatrix} -1 & 4 & -1 \\ -2 & 1 & 5 \\ 4 & -2 & -3 \end{bmatrix}$. Therefore, $v = (1,2,3)^t$ with respect to $B$ is:

$$
P_1^{-1}v = \begin{bmatrix} 4 \\ -\frac{9}{7} \\ -\frac{9}{7} \\ 1 \end{bmatrix}.
$$
(c) Let $P_2 = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ be the basis change matrix from $B'$ to the standard matrix. Then the basis change matrix $P$ from $B$ to $B'$ is nothing but

$$P = P_2^{-1}P_1 = \frac{1}{2} \begin{bmatrix} -1 & 2 & 1 \\ 0 & 0 & 2 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \\ 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & 1 & 0 \\ 0 & 2 & 1 \\ \frac{1}{2} & 0 & 1 \end{bmatrix}.$$  \hfill \Box

**Question 4** (Artin 3.4.2). (a) Determine the base change matrix in $\mathbb{R}^2$, when the old basis is the standard basis $E = (e_1, e_2)$ and the new basis is $B = (e_1 + e_2, e_1 - e_2)$.

(b) Determine the base change matrix in $\mathbb{R}^n$, when the old basis is the standard basis $E$ and the new basis is $B = (e_n, e_{n-1}, \ldots, e_1)$.

(c) Let $B$ be the basis of $\mathbb{R}^2$ in which $v_1 = e_1$ and $v_2$ is a vector of unit length making an angle of 120° with $v_1$. Determine the base change matrix that relates $E$ to $B$.

**Solution.** (a) $P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ is the base change matrix from $B$ to $E$, so its inverse $P^{-1} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ is the desired base change matrix from $E$ to $B$.

(b) Viewing $e_1, \ldots, e_n$ as column vectors, the $n \times n$ matrix $P = [e_n \; e_{n-1} \; \ldots \; e_1]$ is the base change matrix from $B$ to $E$. Hence, the desired matrix is its inverse $P^{-1}$, but one can observe that $P^{-1} = P$. Hence, $P$ is the base change matrix from $E$ to $B$.

(c) Since $v_2 = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, the base change matrix from $E$ to $B$ is $\begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} \end{bmatrix}^{-1} = \begin{bmatrix} 1 & \frac{\sqrt{3}}{3} \\ 0 & \frac{2}{3} \sqrt{3} \end{bmatrix}$. //

**Question 5** (Artin 3.4.3). Let $B = (v_1, \ldots, v_n)$ be a basis of a vector space $V$. Prove that one can get from $B$ to any other basis $B'$ by a finite sequence of steps of the following types:

(i) Replace $v_i$ by $v_i + av_j$, $i \neq j$, for some $a$ in $F$.

(ii) Replace $v_i$ by $cv_i$ for some $c \neq 0$.

(iii) Interchange $v_i$ and $v_j$.

**Proof.** First, note that it suffices to show one can get the standard basis $E$ of $V \cong F^n$ from any basis $B$ using the steps (i–iii), since all of (i–iii) are reversible. Regarding $v_1, \ldots, v_n$ as column vectors, we get a matrix $P = [e_1 \; e_2 \; \ldots \; e_n]$, which is the basis change matrix from $B$ to $E$. Since $P$ is invertible (Question 2), we can decompose $P$ as a product of elementary column matrices, each of which corresponds to one of the steps (i–iii). This concludes the proof. \hfill \Box

**Question 6** (Artin 3.5.2). Let $A$ be a real $n \times n$ matrix. Prove that there is an integer $N$ such that $A$ satisfies a nontrivial polynomial relation $A^N + c_{N-1}A^{N-1} + \ldots + c_1A + c_0 = 0$.

**Proof.** The idea is to regard $M_n(\mathbb{R})$, the set of $n \times n$ real matrices, as a $n^2$-dimensional vector space, and to consider the following set of $n^2 + 1$ vectors:

$$S = \{I, A, A^2, \ldots, A^{n^2}\} \subset M_n(\mathbb{R}).$$

Since $M_n(\mathbb{R})$ is $n^2$-dimensional, it follows that $S$ is linearly dependent. Therefore there exist $a_0, \ldots, a_{n^2} \in \mathbb{R}$ with $(a_0, \ldots, a_{n^2}) \neq (0, \ldots, 0)$ such that $\sum_{i=0}^{n^2} a_iA^n = 0$. Say $N \in \{0, \ldots, n^2\}$ is the largest index such that $a_N \neq 0$. Note we can further assume $N > 0$. Then simply letting $c_i := a_i/a_N$ for $i = 0, \ldots, N - 1$, we get

$$A^N + c_{N-1}A^{N-1} + \ldots + c_1A + c_0 = 0.$$ \hfill \Box