TAYLOR CONDITIONS ON VARIETIES OVER FINITE FIELDS

by

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Abstract

In Chapter 1, we provide some background to the topics discussed throughout the dissertation, including the method of the closed point sieve employed in Poonen's Bertini theorem.

In Chapter 2, we extend Poonen's Bertini theorem over finite fields to Taylor conditions arising from locally free quotients of the sheaf of differentials on projective space. This is motivated by a result of Bilu and Howe in the motivic setting that allows for significantly more general Taylor conditions.

In Chapter 3, we provide a framework that abstracts several instances of implementations of Poonen's closed point sieve.

Chapter 4 consists of joint work with Sean Howe. We formulate an abstract notion of equidistribution for families of λ-probability spaces parameterized by admissible $\mathbb{Z}$-sets. Under the assumption of equidistribution, we show that the σ-moment generating functions of certain infinite sums of random variables can be computed as motivic Euler products. Combining this result with earlier generalizations of Poonen's sieve, we compute the asymptotic Λ-distributions for several natural families of function field L-functions and zeta functions.

Contents

Glossary of Notations

$C(V, W(\mathbb{C}))$	$\mathbb{Z}$-equivariant functions from an admissible $\mathbb{Z}$-set V to $W(\mathbb{C})$
Exp_σ	plethystic exponential function
$\mathbb{F}_q$	finite field with q elements, q a prime power
$H^i(X, \mathscr{F})$	i^{th} sheaf cohomology group of $\mathscr{F}$
$\mathscr{I}_Y$	ideal sheaf of a closed subscheme Y
$\mathcal{L}(\chi, t)$	L-function of a character χ
Λ_A	ring of symmetric functions over a ring A
Λ	ring of symmetric functions over $\mathbb{Z}$
$\Lambda_A^\wedge$	completion of Λ_A with respect to degree filtration
Log_σ	plethystic logarithm function
$\mathscr{O}_X$	structure sheaf of X
$\Omega^1_{X/S}$	sheaf of differentials of X over S
$\mathbb{P}^n_k$	projective n-space over a field k
$\mathscr{P}^r_{X/S}(\mathscr{F})$	sheaf of r-principal parts of $\mathscr{F}$ on X over S
$\mathrm{Sym}^r_{\mathscr{O}_X}(\mathscr{F})$	r^{th} symmetric power of an $\mathscr{O}_X$-module $\mathscr{F}$
$\mathscr{T}\!or_i(\mathscr{F}, \mathscr{G})$	i^{th} Tor sheaf of $\mathscr{O}_X$-modules $\mathscr{F}$ and $\mathscr{G}$
$W(A)$	ring of big Witt vectors of a ring A
ζ_X	local Zeta function of a finite type $\mathbb{F}_q$-scheme X

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Chapter 1

Introduction and background

1.1 Classical Bertini

As many works in this area begin, let us recall a version of Eugenio Bertini's famous theorem:

Theorem 1.1.1 (Classical Bertini)**.** *Let X be a smooth projective subvariety of $\mathbb{P}^n_k$ over a field k. Let $(\mathbb{P}^n_k)^*$ be the dual projective space. Call a hyperplane $H \in (\mathbb{P}^n_k)^*$* good *if $H \cap X$ is smooth and and does not contain X. Then a general element $H \in (\mathbb{P}^n_k)^*$ is good.*

Recall that by a *general element* of $(\mathbb{P}^n_k)^*$, we mean the set of such elements form an open, dense subset of $(\mathbb{P}^n_k)^*$.

As a corollary, if k is algebraically closed, then "almost all" hyperplanes defined over k are good. Furthermore, if k is not necessarily algebraically closed but is infinite, one can always find a good hyperplane defined over k.

Remark 1.1.2. There are various stronger versions of Theorem 1.1.1 stated in terms of the base locus of a linear system on X. Many of these already fail when k is not algebraically closed, or when k is algebraically closed but $\operatorname{char}(k) > 0$. As we are interested in finite fields, our comparison is limited to the version that works in the broadest setting.

For a wonderful history of Bertini's life and theorems, the reader is highly encouraged to consult [Kle98].

1.2 The situation over finite fields

If k is finite, then the set of k-hyperplanes in $\mathbb{P}^n_k$ is also finite; this does not give much room to work with, and things can go very wrong from a classical perspective.

1.2.1 Failure of classical Bertini

In [Kat99], Katz gave an example of a smooth projective hypersurface over $\mathbb{F}_q$ that has *no* smooth $\mathbb{F}_q$-hyperplane sections.

Example 1.2.1 ([Kat99, Question 10])**.** Let $\mathrm{Hyp}(2n+1,q)$ be the smooth hypersurface in $\mathbb{P}^{2n+1}_{\mathbb{F}_q}$ defined by

$$\sum_{i=0}^{n}(x_i y_i^q - x_i^q y_i)$$

where $x_0,\dots,x_n,y_0,\dots,y_n$ are the homogeneous coordinates on $\mathbb{P}^{2n+1}$. Then there is no $\mathbb{F}_q$-hyperplane H such that $X\cap H$ is smooth.

Katz proves this by showing that

(a) $\mathrm{Hyp}(2n+1,q)(\mathbb{F}_q)=\mathbb{P}^{2n+1}(\mathbb{F}_q)$, and

(b) $\mathrm{Hyp}(2n+1,q)$ is isomorphic to its own dual variety.

By definition of the dual variety, this means that every $\mathbb{F}_q$-hyperplane in $\mathbb{P}^{2n+1}$ is tangent to $\mathrm{Hyp}(2n+1,q)$, hence does not intersect it smoothly.

So Theorem 1.1.1 fails over $\mathbb{F}_q$. Katz went on to ask ([Kat99, Question 13]) if some version of Bertini can be salvaged by considering not only hyperplanes but hypersurfaces of degree d sufficiently large. This was answered affirmatively by Gabber in [Gab01] when d is divisible by $p=\mathrm{char}(\mathbb{F}_q)$. Around the same time, though, Poonen proved a much stronger result.

Using a technique called the *closed point sieve*, Poonen showed that, as $d\to\infty$, the probability that a hypersurface intersects X smoothly factors over the probabilities that it is smooth at every closed point x of X.

Definition 1.2.2. Let S_d be the polynomials of degree d in variables $x_0,\dots,x_n$ over $\mathbb{F}_q$, identified with $H^0(\mathbb{P}^n_{\mathbb{F}_q},\mathcal{O}_{\mathbb{P}^n}(d))$. Let $\mathcal{P}_d$ be a subset of S_d. We write

$$\mathrm{Prob}(f\in\mathcal{P}_d):=\frac{\#\mathcal{P}_d}{\#S_d}.$$

Theorem 1.2.3 ([Poo04, Theorem 1.1])**.** *Let X be a smooth quasiprojective subscheme of $\mathbb{P}^n$ of dimension $m\geq 0$ over $\mathbb{F}_q$. Define*

$$\mathcal{P}_d:=\left\{f\in S_d \;\middle|\; H_f\cap U \text{ is smooth of dimension } m-1\right\}.$$

Then

$$\lim_{d\to\infty}\mathrm{Prob}(f\in\mathcal{P}_d)=\prod_{\text{closed } x\in X}\left(1-q^{-(m+1)\deg(x)}\right)=\zeta_X(m+1)^{-1}$$

where ζ_X is the local zeta function of X.

This is a direct geometric analogue of the classical result in number theory that the probability that an integer is square-free is

$$\prod_{p \text{ prime}} (1 - p^{-2}) = \zeta(2)^{-1} = \frac{6}{\pi^2}$$

where here ζ is the Riemann zeta function. In fact, assuming the *abc* conjecture, Poonen gives a shared generalization of Theorem 1.2.3 and this classical result in [Poo04, Theorem 5.1] for quasiprojective schemes over $\mathbb{Z}$.

Before discussing the proof of Theorem 1.2.3, we give a brief exposition of local zeta functions.

1.2.2 Zeta functions

Definition 1.2.4. Let X be a scheme of finite type over $\mathbb{F}_q$. The *local zeta function* of X is

$$\zeta_X(s) = \exp\left(\sum_{m \geq 1} \frac{\#X(\mathbb{F}_{q^m})}{m} q^{-ms}\right).$$

It has an Euler product

$$\zeta_X(s) = \prod_{\text{closed } P \in X} (1 - q^{-s \deg P})^{-1}.$$

By the Weil conjectures (now theorems), ζ_X is a rational function of q^{-s}. For simple X it is possible to compute ζ_X directly using the definition. See [Sil09, Example 2.1] for the example of $\mathbb{P}^n$.

Example 1.2.5. (a) $\zeta_{\mathbb{A}^n_{\mathbb{F}_q}}(s) = \dfrac{1}{1 - q^{n-s}}$

(b) $\zeta_{\mathbb{P}^n_{\mathbb{F}_q}}(s) = \dfrac{1}{(1 - q^{-s})(1 - q^{1-s}) \cdots (1 - q^{n-s})}$

When X is an elliptic curve over $\mathbb{F}_q$, it is a consequence of the Grothendieck-Lefschetz trace formula (proved by Weil for curves) that

$$\zeta_X(s) = \frac{1 - aq^{-s} + q^{1-2s}}{(1 - q^{-s})(1 - q^{1-s})} \tag{1.2.1}$$

where $a = 1 + q - \#X(\mathbb{F}_q)$. See [Sil09, V.2] for a discussion and proof.

Example 1.2.6. Suppose $p \neq 2, 3$ and $p \equiv 2 \pmod 3$. Let E be the elliptic curve over $\mathbb{F}_p$ with Weierstrass form $y^2 = x^3 - B$, B not divisible by p. One can easily show $\#E(\mathbb{F}_p) = p + 1$, so $a = 1 + p - (p + 1) = 2$. By Equation (1.2.1), $\zeta_E(s) = \frac{1 - 2q^{-s} + q^{1-2s}}{(1 - q^{-s})(1 - q^{1-s})}$.

More generally, for $p \neq 2, 3$, any supersingular elliptic curve over $\mathbb{F}_p$ has $p + 1$ points in $\mathbb{F}_p$, hence has the same zeta function as E above.

1.2.3 Closed point sieve

Consider the standard proof that the density of square-free integers is $\prod_p(1-p^{-2})$. For finitely many primes $p_1,\dots,p_s$, the Chinese remainder theorem says that the probability of not being divisible by p_i^2 for $i=1,\dots,s$ is $\prod_{i=1}^s(1-p_i^{-2})$, i.e. the probabilities are independent. But this breaks down when considering infinitely many primes. The difficult part of the proof is showing that the error term vanishes as all primes are included in the computation, meaning showing

$$\lim_{e\to\infty}\lim_{d\to\infty}\frac{\#\left\{\, n\le d \;\middle|\; p^2|n \text{ for some } p>e \,\right\}}{d}=0.$$

Poonen applied this idea to the closed points of a variety. For f homogeneous, $H_f\cap X$ is smooth at a closed point $x\in X$ if and only if a dehomogenization of f at x does not vanish in $\mathscr{O}_{X,x}/\mathfrak{m}_x^2$; equivalently, if and only if the degree zero and one Taylor coefficients of this dehomogenization are not all zero. If X has dimension m, then this is $m+1$ linear conditions over the residue field $\kappa(x)$. Writing $\deg(x):=[\kappa(x):\mathbb{F}_q]$, this means the probability that $H_f\cap X$ is smooth at x is $1-q^{-(m+1)\deg(x)}$. As with square-free integers, these conditions are independent at finitely many points, but not at infinitely many. The difficult part of the proof is showing that the error term (when $d\gg\deg(x)\gg 1$) vanishes.

In fact, Poonen gives stronger versions of Theorem 1.2.3 that allow for controlling the Taylor expansions of f at finitely and infinitely many points ([Poo04, Theorems 1.2 and 1.3]).

1.3 Organization

This thesis has three chapters of new material, all with their own introductory sections describing the motivation for the results therein. At their core, all are motivated by Poonen's Bertini theorem and method of the closed point sieve.

Chapter 2 extends Poonen's Bertini to more general Taylor conditions arising from locally free quotients of the sheaf of differentials of projective space, including with prescribed Taylor expansions at infinitely many points. This allows one to compute more exotic probabilities even related to non-smooth varieties (see Example 2.5.3).

Chapter 3 attempts to unify various implementations of the closed point sieve in the literature into a generally applicable result (Proposition 3.1.3).

Chapter 4 is joint work with Sean Howe. Using Poonen's Bertini as the paradigmatic example, we develop a notion of equidistribution in the setting of admissible $\mathbb{Z}$-sets and show that under this assumption, σ-moment generating functions of certain infinite sums of random variables can be computed as motivic Euler products. We then use this to compute asymptotic Λ-distributions of families of L-functions and zeta functions.

Chapter 2

Taylor conditions on varieties over finite fields

2.1 Introduction

For X a smooth quasiprojective subscheme of $\mathbb{P}^n$ over a finite field $\mathbb{F}_q$, Poonen showed in [Poo04] the existence of smooth hypersurface sections of X and computed the asymptotic density of smooth hypersurface sections to be $\zeta_X(\dim X + 1)^{-1}$, where ζ_X is the zeta function of X. He also allowed for prescribing the first few coefficients of the Taylor expansions of hypersurfaces at finitely many points. It is natural to extend the problem to more general conditions on the Taylor expansions. As far as the author knows, questions like the following are not within the scope of Poonen's theorem or its existing generalizations[1].

Question 2.1.1. Assume $\mathrm{char}(\mathbb{F}_q) \neq 2$. Choose a finite, reduced, degree 4 subscheme Y of $\mathbb{P}^2_{\mathbb{F}_q}$ whose points are geometrically in general position. Let $\iota : X \hookrightarrow \mathbb{P}^2_{\mathbb{F}_q}$ be a curve whose geometric points are in general position with the points of Y. For each closed point $x \in X$, there is a unique smooth conic C_x passing through the four points and X. What is the probability that a random plane curve intersects C_x transversely at x for each closed point $x \in X$?

This question is answered in Example 2.5.3 and requires considering Taylor conditions arising from locally free quotients of the sheaf of differentials on projective space. Such Taylor conditions are addressed in the following theorem which is the main result of this chapter. See Section 2.2.1 for notation and the definition of the sheaf of principal parts $\mathscr{P}^1$.

Theorem A. *Let X be a quasiprojective subscheme of $\mathbb{P}^n_{\mathbb{F}_q}$ of dimension m with locally closed embedding ι. Let $\mathscr{Q}$ be a locally free quotient of $\iota^*\Omega^1_{\mathbb{P}^n}$ of rank $\ell \geq m$, and let $\mathscr{K}$ denote the kernel*

[1] These include [BK12], [EW15], [GK23], [Gun17], [Poo08], and [Wut14].

of $\iota^*\Omega^1_{\mathbb{P}^n} \twoheadrightarrow \mathscr{Q}$. *For each* d, *define*

$$\mathscr{E}_d := \left(\iota^*\mathscr{P}^1(\mathscr{O}_{\mathbb{P}^n}(d))\right)/\mathscr{K}(d)$$

where we view $\mathscr{K}(d)$ *as a subsheaf of* $\iota^*\mathscr{P}^1(\mathscr{O}_{\mathbb{P}^n}(d))$ *via the exact sequence*

$$0 \longrightarrow \iota^*\Omega^1_{\mathbb{P}^n}(d) \longrightarrow \iota^*\mathscr{P}^1(\mathscr{O}_{\mathbb{P}^n}(d)) \longrightarrow \mathscr{O}_X(d) \longrightarrow 0.$$

This defines a 1*-infinitesimal Taylor condition* $\mathcal{T}_d$ *on* $\mathbb{P}^n$ *such that at each closed point* x, $\mathcal{T}_{d,x} \subseteq \mathscr{O}_{\mathbb{P}^n}(d)_x/\mathfrak{m}_x^2$ *is given by not vanishing in the fiber of* $\mathscr{E}_d$ *at* x. *By convention,* $\mathcal{T}_d$ *is always satisfied if* $x \notin X$.

Define

$$\mathcal{P}_d := \left\{ f \in S_d \;\middle|\; f \text{ satisfies } \mathcal{T}_d \text{ at all closed } x \in \mathbb{P}^n \right\}.$$

Then

$$\lim_{d\to\infty} \operatorname{Prob}(f \in \mathcal{P}_d) = \prod_{\text{closed } x\in X} \left(1 - q^{-(\ell+1)\deg(x)}\right) = \zeta_X(\ell+1)^{-1}.$$

Note that for X smooth, taking $\mathscr{Q} = \Omega^1_X$ recovers Poonen's Bertini theorem. Regarding Question 2.1.1, we will define a suitable sheaf $\mathscr{Q}$ in Example 2.5.3 whose fiber at a closed point x is the cotangent space of C_x at x.

Following Poonen, we will prove Theorem A as a special case of the following more general theorem that allows one to prescribe the first few Taylor expansions at finitely many points.

Theorem B. *Let* X *be a quasiprojective subscheme of* $\mathbb{P}^n_{\mathbb{F}_q}$ *and* Z *a finite subscheme of* $\mathbb{P}^n$. *Fix a subset* $T \subseteq H^0(Z, \mathscr{O}_Z)$. *On each connected component* Z_i *of* Z, *fix a nonvanishing coordinate* x_{j_i}. *For* $f \in S_d$, *write* $f|_Z$ *for the element of* $H^0(Z, \mathscr{O}_Z)$ *that on each* Z_i *equals the restriction of* $x_{j_i}^{-d} f$ *to* Z_i.

Assume $U := X - (Z \cap X)$ *has dimension* m *with locally closed embedding* $\iota : U \hookrightarrow \mathbb{P}^n$. *For a locally free quotient* $\mathscr{Q}$ *of* $\iota^*\Omega^1_{\mathbb{P}^n}$ *of rank* $\ell \geq m$, *define* $\mathscr{E}_d$ *and* $\mathcal{T}_d$ *as in Theorem A.*

Define

$$\mathcal{P}_d := \left\{ f \in S_d \;\middle|\; f \text{ satisfies } \mathcal{T}_d \text{ at all closed } x \in \mathbb{P}^n - Z \text{ and } f|_Z \in T \right\}.$$

Then

$$\lim_{d\to\infty} \operatorname{Prob}(f \in \mathcal{P}_d) = \frac{\#T}{\#H^0(Z, \mathscr{O}_Z)} \zeta_U(\ell+1)^{-1}.$$

The proof is an adaptation of Poonen's original proof; the main innovation is observing that the Taylor condition parameterized by X need not have anything to do with properties of X.

Again following Poonen, we will prove a stronger version of Theorem B that allows us to impose Taylor conditions of arbitrary order at infinitely many points so long as the conditions are no stronger than nonvanishing in locally free quotients of the sheaf of principal parts relative to a finite set of varieties.

Theorem C. *Let $X_1,\dots,X_u$ be quasiprojective subschemes of $\mathbb{P}^n_{\mathbb{F}_q}$ of dimensions $\dim X_i = m_i$ with locally closed embeddings $\iota_1,\dots,\iota_u$, respectively. For each i, let $\mathcal{Q}_i$ be a locally free quotient of $\iota_i^*\Omega^1_{\mathbb{P}^n}$ of rank $\ell_i \geq m_i$. Define the sheaves $\mathcal{E}_{i,d}$ and Taylor conditions $\mathcal{T}_{i,d}$ as in Theorem A.*

For each closed point $x \in \mathbb{P}^n$, fix a positive integer M_x, a nonvanishing coordinate x_j, and a subset $A_x \subseteq \mathcal{O}_{\mathbb{P}^n,x}/\mathfrak{m}_x^{M_x}$. For $f \in S_d$, write $f|_x$ for the image of $x_j^{-d} f$ in $\mathcal{O}_{\mathbb{P}^n,x}/\mathfrak{m}_x^{M_x}$. Assume that the sets A_x have been chosen so that for all but finitely many x, $f|_x \in A_x$ whenever $f \in S_d$ satisfies $\mathcal{T}_{i,d}$ at x for all i.

Define

$$\mathcal{P}_d := \left\{ f \in S_d \;\middle|\; f|_x \in A_x \text{ for all closed } x \in \mathbb{P}^n \right\}.$$

Then

$$\lim_{d\to\infty} \operatorname{Prob}(f \in \mathcal{P}_d) = \prod_{\text{closed } x\in X} \frac{\# A_x}{\#\mathcal{O}_{\mathbb{P}^n,x}/\mathfrak{m}_x^{M_x}}.$$

2.1.1 Motivation

Theorems A, B, and C are motivated by the more general Taylor conditions considered by [BH21] in the motivic setting, i.e., in the Grothendieck ring of varieties. There the authors ask if an arithmetic analog of the following theorem holds over $\mathbb{F}_q$ (see the paper for notation):

Theorem ([BH21, Theorem B])**.** *Fix $f: X \to S$, a proper map of varieties over a field K, $\mathcal{F}$ a coherent sheaf on X, $\mathcal{L}$ a relatively ample line bundle on X, and $r, M \geq 0$. Then, there is an $\epsilon > 0$ such that as T ranges over all r-infinitesimal Taylor conditions on $\mathcal{F}(d) = \mathcal{F} \otimes \mathcal{L}^d$ with M-admissible complement,*

$$\frac{[\mathbb{V}(f_*\mathcal{F}(d))^{T\text{-everywhere}}]}{[\mathbb{V}(f_*\mathcal{F}(d))]} = \prod_{x\in X/S} \left(1 - \frac{[T^c]_x}{[\mathbb{V}(\mathcal{P}^r_{/S}\mathcal{F}(d))]_x} t\right)\Bigg|_{t=1} + O(\mathbb{L}^{-\epsilon d})$$

in $\widehat{\widetilde{\mathcal{M}}}_X$.

For Bilu and Howe, a Taylor condition is just a constructible subset of the sheaf of principal parts (viewed as a scheme) and the M-admissible condition ensures the motivic Euler product converges. In the arithmetic setting, we also need a good notion of “admissibility” for a Taylor condition such that the probability that the condition is satisfied everywhere factors into the local probabilities at closed points. A counterexample to the most general such Taylor conditions is given in Example 2.3.1, suggesting more structure, possibly algebraic as in Theorem A, is necessary.

2.1.2 Organization

In Section 2.2 we set up our notation and give some properties of the sheaf of principal parts. Section 2.3 contains a counterexample for the most general Taylor conditions. In Section 2.4 we prove Theorems A, B, and C, and in Section 2.5 we give some applications.

2.2 Notation and definitions

Throughout, let q be a power of a prime p and $\mathbb{F}_q$ the field with q elements. Let $S = \mathbb{F}_q[x_0, \dots, x_n]$ and identify $S_d := H^0(\mathbb{P}^n_{\mathbb{F}_q}, \mathscr{O}(d))$ with degree d homogeneous polynomials in S. Let $A = \mathbb{F}_q[x_1, \dots, x_n]$ and $A_{\le d}$ the polynomials in A of degree at most d. For a closed point

Notation 2.2.1. Let $\mathscr{F}$ be an $\mathscr{O}_X$-module on a locally ringed space X. Let $i : x \hookrightarrow X$ be the inclusion of a point. We write $\mathscr{F}|_x$ for the *fiber* of $\mathscr{F}$ at x, i.e. the $\kappa(x)$-vector space

$$\mathscr{F}|_x = H^0(x, i^*\mathscr{F}) = \mathscr{F}_x \otimes_{\mathscr{O}_{X,x}} \kappa(x).$$

Notation 2.2.2. For X a locally ringed space and $x \in X$, we write $x^{(r)}$ for the r^{th} infinitesimal neighborhood of x, i.e. $x^{(r)} = \operatorname{Spec}(\mathscr{O}_{X,x}/\mathfrak{m}_x^{r+1})$.

Let $j : x^{(r)} \hookrightarrow X$ be the inclusion and $\mathscr{F}$ an $\mathscr{O}_X$-module. We write the restriction of $\mathscr{F}$ to $x^{(r)}$ as

$$\mathscr{F}|_{x^{(r)}} = H^0(x^{(r)}, j^*\mathscr{F}) = \mathscr{F}_x \otimes_{\mathscr{O}_{X,x}} \mathscr{O}_{X,x}/\mathfrak{m}_x^{r+1}.$$

Definition 2.2.3. Let $\mathscr{F}$ be a coherent sheaf on a proper $\mathbb{F}_q$-scheme X. An *r-infinitesimal Taylor condition on $\mathscr{F}$ at a closed point $x \in X$* is a subset

$$\mathcal{T}_x \subseteq \mathscr{F}|_{x^{(r)}}.$$

An *r-infinitesimal Taylor condition $\mathcal{T}$ on $\mathscr{F}$* is a choice of an r-infinitesimal Taylor condition $\mathcal{T}_x$ at x on $\mathscr{F}$ for each closed point x.

We say that a global section $s \in H^0(X, \mathscr{F})$ *satisfies $\mathcal{T}$ at* $x \in X$ if its image in $\mathscr{F}|_{x^{(r)}}$ lies in $\mathcal{T}_x$, and *satisfies $\mathcal{T}$* if it satisfies $\mathcal{T}$ at every closed point $x \in X$.

Definition 2.2.4. Let $\mathscr{F}$ be a coherent sheaf on a proper $\mathbb{F}_q$-scheme X. For a subset $\mathcal{P}$ of the finite dimensional $\mathbb{F}_q$-vector space $H^0(X, \mathscr{F})$, denote by $\operatorname{Prob}(s \in \mathcal{P})$ the probability that a random uniformly distributed global section s of $\mathscr{F}$ belongs to $\mathcal{P}$, i.e.,

$$\operatorname{Prob}(s \in \mathcal{P}) := \frac{\#\mathcal{P}}{\#H^0(X, \mathscr{F})}.$$

Remark 2.2.5. The definition above differs from that of [EW15]. When they write $\mathrm{Prob}(s \in \mathcal{P})$, they mean (in our notation) $\lim_{d\to\infty} \mathrm{Prob}(s_d \in \mathcal{P}_d)$ where for each $d \geq 0$, $\mathcal{P}_d \subseteq H^0(X, \mathscr{F}(d))$ and s_d is a uniform random global section of $\mathscr{F}(d)$.

Remark 2.2.6. Our definition of a Taylor condition assumes X is proper over $\mathbb{F}_q$ so that $H^0(X, \mathscr{F})$ is a finitely generated $\mathbb{F}_q$-vector space. This does not contradict allowing quasiprojective X in Theorem A since there, the Taylor condition is actually on $\mathbb{P}^n$.

2.2.1 Sheaves of principal parts

We recall the definition of sheaves of principal parts and collect some of their relevant properties. These sheaves were introduced by Grothendieck in [Gro67, §16] and have been the object of intermittent study since; recently, they've received some revived interest in commutative algebra in the study of higher order differential operators[2]. Good resources on the subject include the original work in EGA IV, [Ben70, III, §2], [EH16, §7.2], [LT95, §4], and [Per95, Appendix A].

Definition 2.2.7. Let $X \to S$ be a morphism of schemes and $\mathscr{F}$ an $\mathscr{O}_X$-module. Let $\Delta^{(r)}$ be the r-th infinitesimal neighborhood of the diagonal Δ in $X \times_S X$ and let $\delta^{(r)} : \Delta^{(r)} \to X \times_S X$ be the canonical morphism. Denote by $\pi_1, \pi_2 : X \times_S X \to X$ the corresponding projections and set $p = \pi_1 \circ \delta^{(r)}$ and $q = \pi_2 \circ \delta^{(r)}$. The *sheaf of r-th order principal parts of $\mathscr{F}$ on X over S* is

$$\mathscr{P}^r_{X/S}(\mathscr{F}) := p_*(q^*\mathscr{F}).$$

By definition this is an $\mathscr{O}_X$-module. If S is clear from context, we write $\mathscr{P}^r_X(\mathscr{F})$ for $\mathscr{P}^r_{X/S}(\mathscr{F})$; if X is also clear, we write $\mathscr{P}^r(\mathscr{F})$.

References given below are not necessarily the original source of the result.

Lemma 2.2.8 ([Gro67, Proposition 16.7.3])**.** *If $\mathscr{F}$ is quasi-coherent (resp. coherent, of finite type, of finite presentation), then $\mathscr{P}^r_{X/S}(\mathscr{F})$ is quasi-coherent (resp. coherent, of finite type, of finite presentation).*

Lemma 2.2.9 ([Gro67, Corollary 16.4.12] and [Ben70, III, Lemma 2.1 and Proposition 2.2])**.** *If $S = \mathrm{Spec}\, k$ for k a field, $\mathscr{F}$ is quasi-coherent, and $x \in X$ is rational over k, then the fiber $\mathscr{P}^r_{X/S}(\mathscr{F})|_x = \mathscr{P}^r_{X/S}(\mathscr{F})_x \otimes_{\mathscr{O}_{X,x}} \kappa(x)$ is canonically isomorphic to $\mathscr{F}_{X,x}/\mathfrak{m}_x^{r+1}$.*

If k is perfect, then the same is true for any closed point $x \in X$.

[2]See, for example, [BJNB19], [DNB22], and [LY25].

Remark 2.2.10. In our notation, Lemma 2.2.9 says that an r-infinitesimal Taylor condition on $\mathscr{F}$ is just a choice of subset of the fiber of $\mathscr{P}^r_{X/k}(\mathscr{F})$ for every closed $x \in X$.

Lemma 2.2.11 ([Per95, A, Proposition 3.4])**.** *If $X \to S$ is differentially smooth (see [Gro67, 16.10]), and $\mathscr{F}$ is locally free on X, then there is an exact sequence of $\mathscr{O}_X$-modules*

$$0 \longrightarrow \mathrm{Sym}^r_{\mathscr{O}_X}(\Omega^1_{X/S}) \otimes_{\mathscr{O}_X} \mathscr{F} \longrightarrow \mathscr{P}^r_{X/S}(\mathscr{F}) \longrightarrow \mathscr{P}^{r-1}_{X/S}(\mathscr{F}) \longrightarrow 0.$$

If X, Y are smooth S-schemes, $f : X \to Y$ is a morphism of S-schemes, and $\mathscr{G}$ is locally free on Y, then there is a map of exact sequences of $\mathscr{O}_X$-modules

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Sym}^r_{\mathscr{O}_X}(f^*\Omega^1_{Y/S}) \otimes_{\mathscr{O}_X} f^*\mathscr{G} & \longrightarrow & f^*\mathscr{P}^r_{Y/S}(\mathscr{G}) & \longrightarrow & f^*\mathscr{P}^{r-1}_{Y/S}(\mathscr{G}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{Sym}^r_{\mathscr{O}_X}(\Omega^1_{X/S}) \otimes_{\mathscr{O}_X} f^*\mathscr{G} & \longrightarrow & \mathscr{P}^r_{X/S}(f^*\mathscr{G}) & \longrightarrow & \mathscr{P}^{r-1}_{X/S}(f^*\mathscr{G}) & \longrightarrow & 0 \end{array}$$

Corollary 2.2.12 ([Per95, A, Proposition 3.3])**.** *In the setting of Lemma 2.2.11, if $\mathscr{F}$ is locally free of rank n, then $\mathscr{P}^r_{X/S}(\mathscr{F})$ is locally free of rank $n \cdot \binom{\dim X + r}{r}$.*

2.3 Counterexamples to most general Taylor conditions

The following example shows that arbitrary set-theoretic constructions of Taylor conditions even on $\mathscr{O}_{\mathbb{P}^n}(d)$, $d \geq 0$, can produce local probabilities whose product is not the asymptotic global probability of the condition being satisfied.

Example 2.3.1 (Diagonal argument)**.** Let $X = \mathbb{P}^n_{\mathbb{F}_q}$ and $\mathscr{F} = \mathscr{O}_{\mathbb{P}^n}$. Both the union of global sections S_d over all $d \geq 0$ and the set of closed points of $\mathbb{P}^n$ are countably infinite; let $f_1, f_2, \ldots$ and $x_1, x_2, \ldots$ be enumerations of them, respectively. For each i, fix an isomorphism $\mathscr{O}_{\mathbb{P}^n}(d)|_{x_i^{(1)}} \cong \mathscr{O}_{\mathbb{P}^n}|_{x_i^{(1)}}$. Define a 1-infinitesimal Taylor condition $\mathcal{T}_d$ on $\mathscr{O}_{\mathbb{P}^n}(d)$ as follows: for each i, identify $\mathscr{O}_{\mathbb{P}^n}(d)|_{x_i^{(1)}}$ with $\mathscr{O}_{\mathbb{P}^n}|_{x_i^{(1)}}$ under the fixed isomorphism and set $\mathcal{T}_{d,x_i}$ to be all of $\mathscr{O}_{\mathbb{P}^n}|_{x_i^{(1)}}$ except the Taylor expansion of f_i (this does not depend on d). Then the local probabilities are $p_{x_i} = 1 - q^{-(n+1)\deg(x_i)}$ and the product over all closed points is $\zeta_{\mathbb{P}^n}(n+1)^{-1}$.

Globally, however, no section $f \in S_{\mathrm{homog}}$ can satisfy this Taylor condition. Indeed, define

$$\mathcal{P}_d = \{f \in S_d \mid f \text{ satisfies } \mathcal{T}_d \text{ at all closed } x \in \mathbb{P}^n\}.$$

By construction, if $f = f_i$ in our enumeration, then $\mathcal{T}_{d,x_i}$ excludes the Taylor expansion of f, so f fails $\mathcal{T}_d$ at x_i. Thus $\mathcal{P}_d = \emptyset$ for all d, and

$$\lim_{d\to\infty} \mathrm{Prob}(f \in \mathcal{P}_d) = 0 \neq \prod_{i=1}^{\infty} p_{x_i} = \zeta_{\mathbb{P}^n}(n+1)^{-1}.$$

Some algebraic nature to the condition is likely necessary in general. In Theorem A, this manifests as "locally free quotients of the sheaf of differentials".

2.4 More general Taylor conditions

We now use Poonen's method of the closed point sieve to prove our main theorems. Throughout this section, let notation be as in Theorem B.

2.4.1 Points of low degree

The following lemma says that for finitely many closed points, the local probabilities are independent.

Lemma 2.4.1 (Points of low degree)**.** *Let $U_{<e}$ be the closed points of U of degree less than e. Define*

$$\mathcal{P}^{\text{low}}_{d,e} := \left\{ f \in S_d \;\middle|\; f \text{ satisfies } \mathcal{T}_d \text{ at all } x \in U_{<e} \text{ and } f|_Z \in T \right\}.$$

Then

$$\lim_{d\to\infty} \operatorname{Prob}(f \in \mathcal{P}^{\text{low}}_{d,e}) = \frac{\#T}{\#H^0(Z,\mathcal{O}_Z)} \prod_{x\in U_{<e}} \left(1 - q^{-(\ell+1)\deg(x)}\right).$$

Proof. Let $U_{<e} = \{x_1,\dots,x_s\}$. By definition, $f \in S_d$ fails $\mathcal{T}_d$ at x_i if and only if it vanishes under the composition

$$S_d \longrightarrow \mathcal{O}_{\mathbb{P}^n}(d)_{x_i}/\mathfrak{m}^2_{x_i} \longrightarrow \mathcal{E}_d|_{x_i}$$

for some $i \in \{1,\dots,s\}$. Thus $\mathcal{P}^{\text{low}}_{d,e}$ consists of the preimage of $T \times \prod_{i=1}^s (\mathcal{E}_d|_{x_i} - \{0\})$ under the composition

$$S_d \longrightarrow H^0(Z,\mathcal{O}_Z(d)) \times \prod_{i=1}^s \mathcal{O}_{\mathbb{P}^n}(d)_{x_i}/\mathfrak{m}^2_{x_i} \longrightarrow H^0(Z,\mathcal{O}_Z) \times \prod_{i=1}^s \mathcal{E}_d|_{x_i}.$$

The first map is surjective for $d \gg 1$ by [Poo04, Lemma 2.1] and the second since $\iota^*\mathscr{P}^1(\mathcal{O}_{\mathbb{P}^n}(d)) \to \mathcal{E}_d$ is surjective and $H^0(Z,\mathcal{O}_Z(d)) \cong H^0(Z,\mathcal{O}_Z)$, so the composition is surjective.

We have a filtration of $\kappa(x_i)$-vector spaces $0 \subset \mathcal{Q}(d)|_{x_i} \subset \mathcal{E}_d|_{x_i}$ whose quotients $\mathcal{Q}(d)|_{x_i}$ and $\mathcal{E}_d|_{x_i}/\mathcal{Q}(d)|_{x_i}$ have dimensions ℓ and 1, respectively, hence $\mathcal{E}_d|_{x_i} - \{0\}$ has size $q^{(\ell+1)\deg(x_i)} - 1$, and the local probability of vanishing is $1 - q^{-(\ell+1)\deg(x_i)}$. As this does not depend on d, the result follows. □

2.4.2 Points of medium degree

Lemma 2.4.2 (Points of medium degree)**.** *For $e > 0$, define*

$$\mathcal{Q}^{\text{med}}_{d,e} := \left\{ f \in S_d \;\middle|\; f \text{ fails } \mathcal{T}_d \text{ at some } x \in U \text{ with } e \le \deg(x) \le \tfrac{d}{\ell+1} \right\}.$$

Then

$$\lim_{e\to\infty} \lim_{d\to\infty} \operatorname{Prob}(f \in \mathcal{Q}^{\text{med}}_{d,e}) = 0.$$

Proof. Let x be a closed point of U with $e \le \deg(x) \le \ell+1$. We have $\dim_{\mathbb{F}_q} \mathscr{E}_d|_x = (\ell+1)\deg(x) \le d$ by assumption. Note the argument in [Poo04, Lemma 2.1] works exactly the same here with the map $S_d \to \mathscr{E}_d|_x$, so this map is surjective and identical reasoning as in [Poo04, Lemma 2.3] shows the fraction of $f \in S_d$ that vanish in $\mathscr{E}_d|_x$ is $q^{-(\ell+1)\deg(x)}$.

Now we follow Poonen's proof of [Poo04, Lemma 2.4]. By [LW54], there is a constant $c > 0$ depending only on U such that $\#U(\mathbb{F}_{q^r}) \le cq^{rm}$. With the result above, this gives

$$\begin{aligned}
\operatorname{Prob}(f \in \mathcal{Q}^{\text{med}}_{d,e}) &\le \sum_{r=e}^{\lfloor d/(\ell+1)\rfloor} (\text{\# of points of degree } r)\cdot q^{-(\ell+1)r} \\
&\le \sum_{r=e}^{\lfloor d/(\ell+1)\rfloor} \#U(\mathbb{F}_{q^r})\cdot q^{-(\ell+1)r} \\
&\le \sum_{r=e}^{\infty} cq^{rm}q^{-(\ell+1)r}
\end{aligned}$$

Since $\ell \ge m$, this converges to $\frac{cq^{e(m-\ell-1)}}{1-q^{m-\ell-1}}$. This is independent of d and goes to zero as e goes to ∞. □

2.4.3 Points of high degree

As usual with proofs using the closed point sieve, showing the contribution from high degree points is negligible is the hardest part of the proof.

Lemma 2.4.3 (Points of high degree)**.** *Define*

$$\mathcal{Q}^{\text{high}}_{d} := \left\{ f \in S_d \;\middle|\; f \text{ fails } \mathcal{T}_d \text{ at some } x \in U_{>\frac{d}{\ell+1}} \right\}.$$

Then $\lim_{d\to\infty} \operatorname{Prob}(f \in \mathcal{Q}^{\text{high}}_{d}) = 0$.

Proof. As in [Poo04, Lemma 2.6], we reduce to the affine case $\iota : U \hookrightarrow \mathbb{A}^n$, also dehomogenizing to identify S_d with $A_{\le d}$.

Consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \iota^*\Omega^1_{\mathbb{A}^n/\mathbb{F}_q} & \hookrightarrow & \iota^*\mathscr{P}^1(\mathscr{O}_{\mathbb{A}^n}) & \longrightarrow & \mathscr{O}_U & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & & & \\ 0 & \longrightarrow & \mathscr{Q} & \hookrightarrow & \mathscr{E} & & & & \end{array}$$

Given a closed point $x \in U$, the map on fibers $\iota^*\Omega^1_{\mathbb{A}^n/\mathbb{F}_q}|_x \to \mathscr{Q}|_x$ is surjective, and every element of $\iota^*\Omega^1_{\mathbb{A}^n/\mathbb{F}_q}|_x \cong \mathfrak{m}_x/\mathfrak{m}_x^2$ is the restriction of some dt to x where t is some element of A. Choose $t_1,\dots,t_\ell \in A$ such that the restrictions of $dt_1,\dots,dt_\ell$ to x map to a $\kappa(x)$-basis of $\mathscr{Q}|_x$. By Nakayama's lemma, the elements $dt_1,\dots,dt_\ell$ in the stalk $(\iota^*\Omega^1_{\mathbb{A}^n/\mathbb{F}_q})_x$ map to an $\mathscr{O}_{U,x}$-basis for $\mathscr{Q}_x$. Call this basis $Q_1,\dots,Q_\ell$ and let $\partial_1,\dots,\partial_\ell$ be the corresponding dual basis.

Now we mimic the proof of [Poo04, Lemma 2.6].

We have

$$\begin{aligned} \mathrm{Hom}_{\mathscr{O}_{U,x}}(\mathscr{Q}_x,\mathscr{O}_{U,x}) &\subset \mathrm{Hom}_{\mathscr{O}_{U,x}}((\iota^*\Omega^1_{\mathbb{A}^n})_x,\mathscr{O}_{U,x}) \\ &= \mathrm{Hom}_{\mathscr{O}_{\mathbb{A}^n,x}}(\Omega^1_{\mathbb{A}^n,x},\mathscr{O}_{U,x}) \\ &= \mathrm{Der}_{\mathbb{F}_q}(\mathscr{O}_{\mathbb{A}^n,x},\mathscr{O}_{U,x}) \end{aligned}$$

where the first inclusion follows since $\mathrm{Hom}_{\mathscr{O}_{U,x}}(-,\mathscr{O}_{U,x})$ is contravariant left exact. Thus we can think of the dual basis elements ∂_i as $\mathbb{F}_q$-derivations $\mathscr{O}_{\mathbb{A}^n,x} \to \mathscr{O}_{U,x}$. Choose $s \in A/I(U)$ with $s(x) \neq 0$ to clear denominators so $D_i = s\partial_i$ is a global derivation $A \to A/I(U)$. We can find a neighborhood N_x of x on which $Q_1,\dots,Q_\ell$ generate $\mathscr{Q}$ and such that $s \in \mathscr{O}_U(N_x)^\times$. As we can cover U with finitely many such N_x, we may assume $U \subset N_x$, and that the $Q_1,\dots,Q_\ell$ generate $\mathscr{Q}$ globally.

Set $\tau = \max_i\{\deg t_i\}$, $\gamma = \lfloor (d-\tau)/p \rfloor$, and $\eta = \lfloor d/p \rfloor$. If $f_0 \in A_{\le d}$, $g_1,\dots,g_\ell \in A_{\le\gamma}$, and $h \in A_{\le\eta}$ are selected uniformly at random, then the distribution of

$$f = f_0 + g_1^p t_1 + \cdots + g_\ell^p t_\ell + h^p$$

is uniform over $A_{\le d}$. We will bound the probability that for such an f, there is a closed point $y \in U_{>d/(\ell+1)}$ where f is zero in the fiber of $\mathscr{E}$ at y. Let 1 be the constant function in $\iota^*\mathscr{P}^1(\mathscr{O}_{\mathbb{A}^n})$, and R its image in $\mathscr{E}$. Then $R, Q_1,\dots,Q_\ell$ are a basis for $\mathscr{E}$, giving a trivialization $\mathscr{E} \cong \mathscr{O}_U^{\ell+1}$. In this trivialization, the map sending a polynomial f to its first order Taylor expansion in $\iota^*\mathscr{P}^1(\mathscr{O}_{\mathbb{A}^n})$ then to $\mathscr{E}$ is given by $(f,\partial_1 f,\dots,\partial_\ell f)$. Thus f is zero in $\mathscr{E}|_y$ if and only if $f(y) = (D_1 f)(y) = \cdots = (D_\ell f)(y) = 0$.

Since $\mathrm{char}\,\mathbb{F}_q = p$, we have

$$\begin{aligned} D_i f &= D_i f_0 + g_1^p D_i t_1 + t_1 p D_i g_1 + \cdots + g_\ell^p D_i t_\ell + t_\ell p D_i g_\ell + pDh \\ &= D_i f_0 + g_i^p s \end{aligned}$$

for $i = 1, \dots, \ell$. By abuse of notation we will consider the $D_i f$ as defining hypersurfaces in $\mathbb{A}^n$ by choosing a lift to A of minimal degree. Define

$$W_i = U \cap \{D_1 f = \cdots = D_i f = 0\}.$$

Claim 1. *For $0 \le i \le \ell - 1$, conditioned on a choice of $f_0, g_1, \dots, g_i$ such that $\dim(W_i) \le m - i$, the probability that $\dim(W_{i+1}) \le m - i - 1$ is $1 - o(1)$ as $d \to \infty$.*

Let $V_1, \dots, V_e$ be the $(m-i)$-dimensional irreducible components of $(W_i)_{\text{red}}$. By Bézout's theorem,

$$e \le (\deg \overline{U})(\deg D_1 f) \cdots (\deg D_i f) = O(d^i)$$

as $d \to \infty$, where $\overline{U}$ is the projective closure of U. As $\dim V_k \ge 1$, there exists a coordinate x_j, depending on k, such that the projection $x_j(V_k)$ has dimension 1.

We want to bound the set

$$G_k^{\text{bad}} := \left\{ g_{i+1} \in A_{\le \gamma} \;\middle|\; D_{i+1} f = D_{i+1} f_0 + g_{i+1}^p s \text{ vanishes identically on } V_k \right\}$$

since for any $g_{i+1} \in G_k^{\text{bad}}$, $V_k \subset W_{i+1}$ and then $\dim(W_{i+1})$ would fail to be $\le m - i - 1$.

If $g, g' \in G_k^{\text{bad}}$, then on V_k,

$$\begin{aligned} 0 &= \frac{g^p s - g'^p s}{s} \\ &= g^p - g'^p \\ &= (g - g')^p \end{aligned}$$

so if G_k^{bad} is nonempty, it is a coset of the subspace of functions in $A_{\le \gamma}$ that vanish on V_k. The codimension of that subspace is at least $\gamma + 1$ since a nonzero polynomial in x_j does not vanish on V_k. Thus the probability that $D_{i+1} f$ vanishes on some V_k is at most $e q^{-(\gamma+1)} = o(1)$ as $d \to \infty$.

Claim 2. *Conditioned on a choice of $f_0, g_1, \dots, g_\ell$ for which W_ℓ is finite, $\operatorname{Prob}(H_f \cap W_\ell \cap U_{>d/(\ell+1)} = \emptyset) = 1 - o(1)$ as $d \to \infty$.*

In fact, we need only show this for $H_f \cap W_m \cap U_{>d/(\ell+1)}$. The same Bézout argument as above shows $\# W_m$ is $O(d^m)$. For a given $y \in W_m$, the set H^{bad} of $h \in A_{\le \eta}$ for which H_f passes through y is either empty or a coset of $\ker(\operatorname{eval}_y : A_{\le \eta} \to \kappa(y))$.

If $\deg(y) > \frac{d}{\ell+1}$, then [Poo04, Lemma 2.5] implies $\frac{\# H^{\text{bad}}}{\# A_{\le \eta}} \le q^{-\nu}$ where $\nu = \min(\eta + 1, \frac{d}{\ell+1})$. Hence

$$\operatorname{Prob}(H_f \cap W_m \cap U_{>d/(\ell+1)} \ne \emptyset) \le \# W_m q^{-\nu} = O(d^m q^{-\nu})$$

which by assumption is $o(1)$ as $d \to \infty$.

Given the two claims, we have

$$\begin{aligned}\lim_{d\to\infty} \operatorname{Prob}\big(\dim W_i = m - i \text{ for all } 1 \le i \le \ell \text{ and } H_f \cap W_m \cap U_{>d/(\ell+1)} = \emptyset\big)\\ = \prod_{i=0}^{m-1} (1 - o(1)) \cdot (1 - o(1))\\ = 1 - o(1).\end{aligned}$$

So the same holds for W_ℓ. But now $H_f \cap W_\ell$ is the subvariety of U defined by failing $\mathcal{T}_d$, so $H_f \cap W_\ell \cap U_{>d/(\ell+1)}$ is the set of points of degree $> \frac{d}{\ell+1}$ where $H_f \cap U$ fails $\mathcal{T}_d$. □

2.4.4 Proofs of Theorems A, B, and C

Proof of Theorem B. We have

$$\mathcal{P}_d \subseteq \mathcal{P}_{d,e}^{\text{low}} \subseteq \mathcal{P}_d \cup \mathcal{Q}_{d,e}^{\text{med}} \cup \mathcal{Q}_d^{\text{high}}$$

so

$$\begin{aligned}\operatorname{Prob}(s \in \mathcal{P}_{d,e}^{\text{low}}) &\ge \operatorname{Prob}(s \in \mathcal{P}_d)\\ &\ge \operatorname{Prob}(s \in \mathcal{P}_{d,e}^{\text{low}}) - \operatorname{Prob}(s \in \mathcal{Q}_{d,e}^{\text{med}}) - \operatorname{Prob}(s \in \mathcal{Q}_d^{\text{high}}).\end{aligned}$$

By Lemmas 2.4.1 to 2.4.3, letting d, then e go to ∞ gives the result. □

Proof of Theorem A. Take $Z = \emptyset$ in Theorem B. □

Proof of Theorem C. The reasoning here is the same as in the proof of [Poo04, Theorem 1.3]. Given a condition on sections no stronger than nonvanishing in the fiber of a single $\mathcal{E}_{i,d}$ at all except finitely many points, the probability of failing this condition goes to 0 as $d \to \infty$ by Lemmas 2.4.2 and 2.4.3. Now considering a condition no stronger than nonvanishing in the fiber of each of $\mathcal{E}_{1,d}, \dots, \mathcal{E}_{u,d}$ at all except finitely many points, the probability of failing is still zero as this is a finite union of sets with probability zero. Thus we can approximate $\mathcal{P}_d$ by the sets $\mathcal{P}_{d,e}^{\text{low}}$ defined by satisfying the condition at points of degree at most e. But now the result follows by [Poo04, Lemma 2.1] and identical reasoning as in the proof of Lemma 2.4.1. □

2.5 Applications

Example 2.5.1 (Poonen's Bertini)**.** To get [Poo04, Theorem 1.1], assume X is smooth and take $\mathcal{Q} = \Omega^1_{X/\mathbb{F}_q}$ in Theorem A. Similarly, [Poo04, Theorem 1.2] follows from Theorem B.

Our Theorem C does not imply [Poo04, Theorem 1.3] since we don't work with the completed local rings, however it does imply the weaker version where, at each point, one only controls the Taylor expansion up to finitely many terms.

Example 2.5.2. Let X be a quasiprojective subscheme of $\mathbb{P}^n_{\mathbb{F}_q}$ of dimension m with locally closed embedding ι and let $\Delta : X \hookrightarrow X \times_{\mathbb{F}_q} \mathbb{P}^n$ be the graph of ι. Suppose $j : Z \hookrightarrow X \times_{\mathbb{F}_q} \mathbb{P}^n$ is a closed embedding such that the projection $\varphi : Z \to X$ is smooth of relative dimension $\ell \geq m$, and such that Δ factors as

$$X \xrightarrow{\alpha} Z \xhookrightarrow{j} X \times \mathbb{P}^n$$

for some morphism $\alpha : X \to Z$.

We have a surjection of sheaves

$$\Omega^1_{X\times\mathbb{P}^n/X} \twoheadrightarrow j_*\Omega^1_{Z/X}$$

which induces a surjection

$$\Delta^*\Omega^1_{X\times\mathbb{P}^n/X} \twoheadrightarrow \Delta^* j_*\Omega^1_{Z/X}.$$

The left side is isomorphic to $\iota^*\Omega^1_{\mathbb{P}^n}$; indeed, let $p : X \times \mathbb{P}^n \to \mathbb{P}^n$ be projection onto the second coordinate. Then by standard base change for the sheaf of differentials,

$$\begin{aligned}\Delta^*\Omega^1_{X\times\mathbb{P}^n/X} &\cong \Delta^* p^*\Omega^1_{\mathbb{P}^n/\mathbb{F}_q}\\ &= (p\circ\Delta)^*\Omega^1_{\mathbb{P}^n/\mathbb{F}_q}\\ &= \iota^*\Omega^1_{\mathbb{P}^n/\mathbb{F}_q}.\end{aligned}$$

Define $\mathcal{Q} = \Delta^* j_*\Omega^1_{Z/X}$. This is locally free: by assumption, $\Delta = j\circ\alpha$ so $\mathcal{Q} = \alpha^* j^* j_*\Omega^1_{Z/X} \cong \alpha^*\Omega^1_{Z/X}$. As φ is smooth of relative dimension ℓ, $\Omega^1_{Z/X}$ is locally free of rank ℓ and thus so is $\mathcal{Q}$.

With $\mathcal{Q}$ as above, define $\mathscr{E}_d$, $\mathcal{T}_d$, and $\mathcal{P}_d$ as in Theorem A. Applying the theorem, we get

$$\lim_{d\to\infty} \operatorname{Prob}(f \in \mathcal{P}_d) = \zeta_X(\ell+1)^{-1}.$$

Example 2.5.3. We now answer Question 2.1.1 as a specific instance of Example 2.5.2. Assume $\operatorname{char}(\mathbb{F}_q) \neq 2$. Choose a finite, reduced, degree 4 subscheme Y of $\mathbb{P}^2_{\mathbb{F}_q}$ whose points are geometrically in general position. Let $\iota : X \hookrightarrow \mathbb{P}^2_{\mathbb{F}_q}$ be a curve whose geometric points are in general position with the points of Y. Then for each closed point $x \in X$, there is a unique smooth conic C_x (defined over $\kappa(x)$) passing through x and each point of Y. Let $j : C \hookrightarrow X \times \mathbb{P}^2$ be the inclusion of the subscheme C parameterizing the data $\{(x, y) \mid x \in X, y \in C_x\}$. Then Δ factors as $j\circ\alpha$ where

α is the diagonal into C, and $\varphi : C \to X$ is smooth of relative dimension 1, so the conditions of the example are satisfied.

Let $f \in S_d$. With $\mathcal{Q}$ defined as in Example 2.5.2, the hypersurface H_f intersects C_x transversely at x if and only if it does not vanish in the fiber of $\mathcal{Q}$ at x. Thus the example above shows the probability that a random plane curve intersects C_x transversely at x for all closed $x \in X$ is $\zeta_X(2)^{-1}$.

Example 2.5.4. Let L be the line at infinity in $\mathbb{P}^2_{\mathbb{F}_q}$; write the homogeneous coordinates on $\mathbb{P}^2$ as x_0, x_1, x_2. In the affine chart $x_0 \neq 0$, choose $\mathbb{F}_q$-points $P_1 = (0,0)$, $P_2 = (0,1)$, $P_3 = (1,0)$, and $P_4 = (1,1)$. Then the lines through pairs of points in P_1, P_2, P_3, P_4 intersect L in four points; set U to be L with these four points removed and $Y := \{P_1, P_2, P_3, P_4\}$. Define C_x as above. By Example 2.5.3, the probability that a random plane curve intersects C_x transversely at x for all $x \in U$ is $\zeta_U(2)^{-1}$. Recall that for a scheme X of finite type over $\mathbb{F}_q$ with closed subscheme Z, we have $\zeta_{X\setminus Z}(s) = \zeta_X(s)/\zeta_Z(s)$. Writing $Z = \{Q_1, Q_2, Q_3, Q_4\}$ for the set of four points removed from L, this implies

$$\zeta_U(2)^{-1} = \frac{\zeta_Z(2)}{\zeta_L(2)} = \frac{(1-q^{-2})^{-4}}{\frac{1}{(1-q^{-2})(1-q^{-1})}} = \frac{1-q^{-1}}{(1-q^{-2})^3}.$$

Chapter 3

Axiomatic approach to Bertini theorems over finite fields

After Poonen's Bertini theorem, numerous analogous and more general Bertini-type results followed. For those generalizations that relate to *Taylor conditions* (see Definition 2.2.3), we provide a framework (Proposition 3.1.3) that axiomatizes the general strategy of the proofs and that can, in theory, be applied to prove similar results.

This chapter is motivated by the results of [CGM86] and [Spr98] that provide powerful axiomatic frameworks for proving Bertini-type theorems over algebraically closed and infinite fields, respectively. Recent work has used the Cumino-Greco-Manaresi framework to prove Bertini theorems in the positive characteristic, algebraically closed setting: F-regular and F-pure singularities ([SZ13]), F-signature and Hilbert-Kunz multiplicity assuming normality ([CRST21]), and Hilbert-Kunz multiplicity along fibers and without assuming normality ([DS22]).

Our framework is not nearly as useful. First, our framework applies only to conditions on the Taylor coefficients of sections; theirs applies to *any* condition that satisfies certain axioms. Second, our result is not particularly useful for proving new theorems since it pushes the difficult part of the proof, showing part (c) of Proposition 3.1.3, down the road.

Further motivation comes from the work of Margaret Bilu and Sean Howe in the motivic setting. In [BH21, Theorem B] (also restated and discussed in Section 2.1.1), a very general class of Taylor conditions were shown to give asymptotic "global probabilities" that factor into the asymptotic "local probabilities" in a suitable localization and completion of the Grothendieck ring of varieties. However, neither the results nor methods are applicable in the point-counting setting. In [BH21, Section 1.3], the authors ask if a similar theorem exists in the point-counting setting. This chapter *does not* provide an answer to this question; it does, we hope, provide a useful framework for thinking about the diverse applications of Poonen's closed point sieve, often appearing ad hoc in the literature.

3.1 The framework

Throughout this chapter, we will use the notation and definitions given in Section 2.2. Before stating the axiomatic framework, we need a lemma.

Lemma 3.1.1. *Let Y be a projective scheme over a field k. If Z is a finite subscheme of Y, and $\mathscr{F}$ is a coherent sheaf on Y, then the map*

$$\phi_d : H^0(Y, \mathscr{F}(d)) \longrightarrow H^0(Z, j^*\mathscr{F}(d))$$

is surjective for $d \gg 0$, where $j : Z \hookrightarrow Y$ is the inclusion morphism.

Proof. We have a short exact sequence

$$0 \longrightarrow \mathscr{I}_Z \longrightarrow \mathscr{O}_Y \longrightarrow j_*\mathscr{O}_Z \longrightarrow 0$$

which, after tensoring with $\mathscr{F}$, induces an exact sequence

$$0 \longrightarrow \mathscr{T}\!or_1(j_*\mathscr{O}_Z, \mathscr{F}) \longrightarrow \mathscr{I}_Z \otimes \mathscr{F} \longrightarrow \mathscr{F} \longrightarrow (j_*\mathscr{O}_Z) \otimes \mathscr{F} \longrightarrow 0.$$

By the projection formula, $(j_*\mathscr{O}_Z) \otimes \mathscr{F} \cong j_* j^*\mathscr{F}$. Set $\mathscr{G} = \operatorname{coker}\big(\mathscr{T}\!or_1(j_*\mathscr{O}_Z, \mathscr{F}) \to \mathscr{I}_Z \otimes \mathscr{F}\big)$. The exact sequence above gives a short exact sequence

$$0 \longrightarrow \mathscr{G} \longrightarrow \mathscr{F} \longrightarrow j_* j^*\mathscr{F} \longrightarrow 0.$$

Twisting by $\mathscr{O}_Y(d)$ gives

$$0 \longrightarrow \mathscr{G}(d) \longrightarrow \mathscr{F}(d) \longrightarrow j_* j^*\mathscr{F}(d) \longrightarrow 0$$

inducing the long exact sequence in cohomology

$$0 \longrightarrow H^0(Y, \mathscr{G}(d)) \longrightarrow H^0(Y, \mathscr{F}(d)) \xrightarrow{\phi_d} H^0(Z, j^*\mathscr{F}(d)) \longrightarrow H^1(Y, \mathscr{G}(d)) \longrightarrow \cdots$$

Thus $\operatorname{coker}(\phi_d)$ is a sub-vector space of $H^1(Y, \mathscr{G}(d))$. Since $\mathscr{G}$ is coherent (see Remark 3.1.2), $H^1(Y, \mathscr{G}(d)) = 0$ for $d \gg 0$ by Serre vanishing ([Har77, Theorem 5.2]). Hence $\operatorname{coker}(\phi_d) = 0$ for $d \gg 0$, i.e. ϕ_d is surjective. □

Remark 3.1.2. The sheaf $\mathscr{G}$ above is indeed coherent. The sheaf $\mathscr{T}or_i(j_*\mathscr{O}_Z, \mathscr{F})$ can be computed as the homology of the complex $\mathscr{L}^\bullet \otimes \mathscr{F}$ where $\mathscr{L}^\bullet \to j_*\mathscr{O}_Z$ is a resolution by finite, locally free $\mathscr{O}_Y$-modules (such a resolution always exists in this setting; see [Har77, Example 6.5.1]). This is coherent since the category of coherent sheaves on a scheme is abelian, so closed under kernels, cokernels, and images. Since $\mathscr{I}_Z \otimes \mathscr{F}$ is coherent, the cokernel $\mathscr{G}$ is also coherent.

Proposition 3.1.3. *Let $\mathscr{F}$ be a coherent sheaf on a projective scheme Y over $\mathbb{F}_q$. For $d \geq 0$, let $V_d := H^0(Y, \mathscr{F}(d))$. For each $d \geq 0$, let $\mathcal{T}_d$ be an r-infinitesimal Taylor condition on $\mathscr{F}(d)$. Fix a function $c : \mathbb{N} \to \mathbb{R}_{\geq 1}$. Define*

$$\mathcal{P}_{e,d}^{\text{low}} := \left\{ s \in V_d \mid s \textit{ satisfies } \mathcal{T}_d \textit{ at all } x \textit{ with } \deg(x) < e \right\}$$
$$\mathcal{Q}_{e,d}^{\text{med}} := \left\{ s \in V_d \mid s \textit{ fails } \mathcal{T}_d \textit{ at some } x \textit{ with } e \leq \deg(x) \leq c(d) \right\}$$
$$\mathcal{Q}_{d}^{\text{high}} := \left\{ s \in V_d \mid s \textit{ fails } \mathcal{T}_d \textit{ at some } x \textit{ with } \deg(x) > c(d) \right\}$$

Suppose that

(a) *For each closed point $x \in Y$,*
$$p_x := \lim_{d\to\infty} \frac{\#\mathcal{T}_{d,x}}{\#\mathscr{F}(d)|_{x^{(r)}}}$$
exists.

(b) $\lim_{e\to\infty} \lim_{d\to\infty} \operatorname{Prob}(s \in \mathcal{Q}_{e,d}^{\text{med}}) = 0$

(c) $\lim_{d\to\infty} \operatorname{Prob}(s \in \mathcal{Q}_{d}^{\text{high}}) = 0$

Then
$$\lim_{d\to\infty} \operatorname{Prob}(s \textit{ satisfies } \mathcal{T}_d) = \prod_{x\in Y} p_x.$$

Proof. First we show independence at finitely many closed points, namely that
$$\lim_{d\to\infty} \operatorname{Prob}(s \in \mathcal{P}_{e,d}^{\text{low}}) = \prod_{\substack{x\in Y\\ \deg(x)<e}} p_x.$$

For a closed point $x \in Y$, let $x^{(r)}$ be the r^{th} infinitesimal neighborhood of x in Y. Define $Z = \bigsqcup x^{(r)}$ where the union is over the finitely many closed points $x \in Y$ of degree $< e$. A section $s \in V_d$ satisfies $\mathcal{T}_d$ at x if and only if its restriction in $H^0(x^{(r)}, j^*\mathscr{F}(d)) = \mathscr{F}(d)|_{x^{(r)}}$ lies in $\mathcal{T}_{d,x}$, where $j : Z \hookrightarrow X$ is the inclusion. Thus $\mathcal{P}_{e,d}^{\text{low}}$ is the preimage of
$$\prod_{\substack{x\in Y\\ \deg(x)<e}} \mathcal{T}_{d,x}$$

under

$$\phi_d : V_d \longrightarrow H^0(Z, j^*\mathscr{F}(d)) \cong \prod_{\substack{x\in Y \\ \deg(x)<e}} \mathscr{F}(d)|_{x^{(r)}}.$$

By Lemma 3.1.1, ϕ_d is surjective for $d \gg 0$, so

$$\lim_{d\to\infty} \operatorname{Prob}(s \in \mathcal{P}^{\text{low}}_{e,d}) = \lim_{d\to\infty} \prod_{\substack{x\in Y \\ \deg(x)<e}} \frac{\#\mathcal{T}_{d,x}}{\#\mathscr{F}(d)|_{x^{(r)}}} = \prod_{\substack{x\in Y \\ \deg(x)<e}} p_x.$$

To finish the proof, the reasoning is the same as in the proofs of [Poo04, Theorem 1.2] and [EW15, Theorem 3.1]. Let $\mathcal{P}_d$ be the set of $s \in V_d$ that satisfy $\mathcal{T}_d$. We have

$$\mathcal{P}_d \subseteq \mathcal{P}^{\text{low}}_{e,d} \subseteq \mathcal{P}_d \cup \mathcal{Q}^{\text{med}}_{e,d} \cup \mathcal{Q}^{\text{high}}_d$$

so

$$\begin{aligned}\operatorname{Prob}(s \in \mathcal{P}^{\text{low}}_{e,d}) &\geq \operatorname{Prob}(s \in \mathcal{P}_d) \\ &\geq \operatorname{Prob}(s \in \mathcal{P}^{\text{low}}_{e,d}) - \operatorname{Prob}(s \in \mathcal{Q}^{\text{med}}_{e,d}) - \operatorname{Prob}(s \in \mathcal{Q}^{\text{high}}_d).\end{aligned}$$

Applying our assumptions and computation above, letting d, then e go to ∞ gives the result. □

Remark 3.1.4. In Proposition 3.1.3, both the function $c(d)$ and splitting the "non-low" degree points into "medium" and "high" degree are not important to the proof. All that matters is that the local probabilities p_x exist and that the probability that a section has a singularity of degree e goes to zero as d, then e go to ∞. We phrase it like this only to match the existing implementation of the closed point sieve.

3.2 Observing the framework in action

Now we go through several examples in the literature, observing how the framework of Proposition 3.1.3 is implicitly applied. For simplicity of exposition we will work with versions of the various theorems that do not impose additional Taylor conditions, however the framework can be easily modified to apply in those settings as well.

3.2.1 Poonen's Bertini

Let $S = \mathbb{F}_q[x_0, \dots, x_n]$, S_d the degree d homogeneous elements of S, and $S_{\text{homog}} = \bigcup_{d\geq 0} S_d$. Set $H_f = \operatorname{Proj}(S/(f)) \subseteq \mathbb{P}^n_{\mathbb{F}_q}$.

We restate Poonen's Bertini theorem given in the introduction:

Theorem 1.2.3 ([Poo04, Theorem 1.1])**.** *Let X be a smooth quasiprojective subscheme of $\mathbb{P}^n$ of dimension $m \geq 0$ over $\mathbb{F}_q$. Define*

$$\mathcal{P}_d := \left\{ f \in S_d \;\middle|\; H_f \cap U \text{ is smooth of dimension } m-1 \right\}.$$

Then

$$\lim_{d\to\infty} \operatorname{Prob}(f \in \mathcal{P}_d) = \prod_{\text{closed } x\in X} \left(1 - q^{-(m+1)\deg(x)}\right) = \zeta_X(m+1)^{-1}$$

where ζ_X is the local zeta function of X.

In the context of Proposition 3.1.3, set $Y = \mathbb{P}^n_{\mathbb{F}_q}$ and $\mathscr{F} = \mathscr{O}_{\mathbb{P}^n}$. For each $d \geq 0$ and closed $x \in \mathbb{P}^n$, define

$$\mathcal{T}_{d,x} := \begin{cases} \left\{ g_x \in \mathscr{O}_{\mathbb{P}^n}(d)_x/\mathfrak{m}_x^2 \;\middle|\; \text{image of } g_x \text{ in } \mathscr{O}_{X,x}/\mathfrak{m}_x^2 \text{ is nonzero} \right\} & x \in X \\ \mathscr{O}_{\mathbb{P}^n,x}(d)/\mathfrak{m}_x^2 & x \notin X \end{cases}$$

Set $c(d) = \frac{d}{m+1}$ and define the subsets $\mathcal{P}^{\text{low}}_{e,d}$, $\mathcal{Q}^{\text{med}}_{e,d}$, and $\mathcal{Q}^{\text{high}}_{d}$ of S_d as in Proposition 3.1.3.

We'll show conditions (a)-(c).

(a) By (noncanonically) untwisting, one sees the terms $\frac{\#\mathcal{T}_{d,x}}{\#\mathscr{F}(d)|_{x^{(1)}}}$ do not depend on d, so certainly their limits p_x exist, and equal $1 - q^{-(m+1)\deg(x)}$ by [Poo04, Lemma 2.2].

(b) This is [Poo04, Lemma 2.4].

(c) This is [Poo04, Lemma 2.6].

Hence the axiomatic framework applies. Note that this example will be subsumed under all examples that follow.

3.2.2 Complete intersections

For $\mathbf{d} = (d_1, \ldots, d_k)$ a tuple of positive integers, write $S_{\mathbf{d}}$ for the product $S_{d_1} \times \ldots \times S_{d_k}$ and identify it with the global sections of $\mathscr{O}_{\mathbb{P}^n}(\mathbf{d}) = \bigoplus_{i=1}^k \mathscr{O}_{\mathbb{P}^n}(d_i)$. For each $\mathbf{f} = (f_1, \ldots, f_k) \in S_{\mathbf{d}}$, write $H_{\mathbf{f}}$ for $H_{f_1} \cap \ldots \cap H_{f_k}$.

Define

$$L(q,m,k) = \prod_{j=0}^{k-1} (1 - q^{-(m-j)})$$

which is the probability that k randomly chosen vectors in $\mathbb{F}_q^m$ are linearly independent.

Theorem 3.2.1 ([BK12, Theorem 1.2 without additional Taylor conditions])**.** *Let X be a smooth quasiprojective subscheme of $\mathbb{P}^n$ of dimension $m \geq 0$ over $\mathbb{F}_q$. Let $\overline{X}$ denote the Zariski closure of*

X in $\mathbb{P}^n$. Choose an integer $k \in \{1, \dots, m+1\}$ and a tuple $\mathbf{d} = (d_1, \dots, d_k)$ of positive integers with $d_1 \le \dots \le d_k$. Define

$$\mathcal{P}_{\mathbf{d}} := \big\{ \mathbf{f} \in S_{\mathbf{d}} \;\big|\; H_{\mathbf{f}} \cap X \text{ is smooth of dimension } m-k \big\}.$$

Then

$$\operatorname{Prob}(\mathbf{f} \in \mathcal{P}_{\mathbf{d}}) = \prod_{x \in X} \big(1 - q^{-k\deg(x)} + q^{-k\deg(x)} L(q^{\deg(x)}, m, k)\big) \\ + O\big((d_1+1)^{-(2k-1)/m} + d_k^m q^{-d_1/\max\{m+1,p\}}\big)$$

where the implied constant is an increasing function of n, m, k, and $\deg(\overline{X})$. Thus letting $d_1, \dots, d_k \to \infty$ such that $d_k^m q^{-d_1/\max\{m+1,p\}} \to 0$, we get the probability converging to the product of the local factors.

Due to limitations in the setup of Proposition 3.1.3, we will only consider that case that $d_1, \dots, d_k \to \infty$ at a uniform constant rate for all k. However, one could modify the statement similar to Definition 4.4.1 to account for nets of Taylor conditions $\mathcal{T}_d$, $d \in I$ for I a directed set. This is done for this example in Section 4.6.1.

Set $Y = \mathbb{P}^n_{\mathbb{F}_q}$ and $\mathscr{F} = \mathscr{O}_{\mathbb{P}^n}(\mathbf{d}_0)$ where $\mathbf{d}_0 = (d_1, \dots, d_k)$ is any fixed tuple of nonnegative integers (again, an artifact of our limited setup). We will consider the twists $\mathscr{F}(d) \cong \bigoplus_{i=1}^k \mathscr{O}_{\mathbb{P}^n}(d_i + d)$ as $d \to \infty$. Note that

$$(d_k + d)^m q^{-(d_1+d)/(m+1)} \to 0 \quad \text{as} \quad d \to \infty$$

so in this setting the error term of Theorem 3.2.1 tends to zero.

For each $d \ge 0$ and closed $x \in \mathbb{P}^n$, let $\mathcal{T}_{d,x}$ be the subset of $\mathscr{F}(d)|_{x^{(1)}} = \mathscr{O}_{\mathbb{P}^n}(\mathbf{d}_0 + d)_x / \mathfrak{m}_x^2$ where, for the image of a tuple $\mathbf{f} = (f_1, \dots, f_k) \in \mathscr{F}(d)|_{x^{(1)}}$ in $\mathscr{O}_X(\mathbf{d}_0 + d)_x/\mathfrak{m}_x^2$, either not all terms vanish, or they all vanish and have linearly independent gradients. Set $c(d) = \frac{d_1+d}{m+1}$ and define the subsets $\mathcal{P}^{\text{low}}_{e,d}$, $\mathcal{Q}^{\text{med}}_{e,d}$, and $\mathcal{Q}^{\text{high}}_d$ of $V_d := H^0(\mathbb{P}^n, \mathscr{F}(d))$ as in Proposition 3.1.3.

(a) As for Poonen's Bertini, the local factors do not depend on d, and equal $1 - q^{-k\deg(x)} + q^{-k\deg(x)} L(q^{\deg(x)}, m, k)$ by [BK12, Corollary 2.2].

(b) By [BK12, Lemma 2.5],

$$\operatorname{Prob}(\mathbf{f} \in \mathcal{Q}^{\text{med}}_{e,d}) \le 2^{m+1} \deg(\overline{X}) k q^{-e(2k-1)}.$$

This bound does not depend on d and goes to 0 as $e \to \infty$.

(c) By [BK12, Corollary 2.7],

$$\operatorname{Prob}(\mathbf{f} \in \mathcal{Q}_d^{\text{high}}) \le k^m n^{2m}(m+1)\deg(\overline{X})^{m+1}(d_k+d)^m q^{-\min\{(d_1+d)/(m+1),(d_1+d)/p\}}$$

which goes to 0 as $d \to \infty$.

3.2.3 Semiample Bertini

Let X be a smooth projective variety (integral but not necessarily geometrically integral) of dimension m over $\mathbb{F}_q$, with a very ample divisor A and a globally generated divisor E. Let π be the map given by the complete linear series on E:

$$\pi : X \xrightarrow{|E|} \mathbb{P}^M.$$

Let $b = \dim \pi(X)$. Define $R_{n,d} := H^0(X, \mathcal{O}_X(nA + dE))$ and write H_f for the corresponding divisor in $|nA + dE|$.

Theorem 3.2.2 ([EW15, Theorem 3.1 without additional Taylor conditions])**.** *With notation as above, set $n_0 := \max\{b(m+1)-1, bp+1\}$. For a closed point $P \in \pi(X)$, define*

$$p_P := \lim_{d\to\infty} \operatorname{Prob}(H_f \textit{ is smooth at all points of } \pi^{-1}(P))$$

and

$$\mathcal{P}_d := \left\{ f \in R_{n,d} \;\middle|\; H_f \cap X \textit{ is smooth} \right\}.$$

Then for all $n \ge n_0$, we have

$$\lim_{d\to\infty} \operatorname{Prob}(f \in \mathcal{P}_d) = \prod_{P \in \pi(X)} p_P.$$

Let $P \in \pi(X)$ and $P^{(1)} = \operatorname{Spec}(\mathcal{O}_{\pi(X),P}/\mathfrak{m}_P^2)$ the first-order infinitesimal neighborhood of P. For $x \in \pi^{-1}(P)$, define similarly $x^{(1)} = \operatorname{Spec}(\mathcal{O}_{X,x}/\mathfrak{m}_x^2)$. Set $X_{P^{(1)}} := X \times_{\pi(X)} P^{(1)}$.

Given a section $f \in R_{n,d}$, H_f is smooth at a closed point $x \in \pi^{-1}(P)$ if and only if f does not vanish under the restriction map

$$R_{n,d} \longrightarrow H^0(x^{(1)}, \mathcal{O}_{x^{(1)}}(nA)) \cong \mathcal{O}_{X,x}/\mathfrak{m}_x^2.$$

where the latter isomorphism is a noncanonical untwisting. This map factors as

$$R_{n,d} \xrightarrow{g} H^0(X_P^{(1)}, \mathcal{O}_{X_P^{(1)}}(nA)) \xrightarrow{\varphi_x} \mathcal{O}_{X,x}/\mathfrak{m}_x^2.$$

In the context of Proposition 3.1.3, set $Y = \pi(X)$ and $\mathscr{F} = \pi_*(\mathscr{O}_X(nA))$, so $\mathscr{F}(d) \cong \pi_*(\mathscr{O}_X(nA + dE))$. There is a natural map ([Har77, III, §11])

$$\mathscr{F}(d)|_{P^{(1)}} = \pi_*(\mathscr{O}_X(nA + dE))_P \otimes_{\mathscr{O}_{\pi(X),P}} \mathscr{O}_{\pi(X),P}/\mathfrak{m}_P^2 \xrightarrow{k} H^0\big(X_{P^{(1)}}, \mathscr{O}_{X_{P^{(1)}}}(nA)\big).$$

Note that g is the composition of the natural restriction map

$$R_{n,d} = H^0(\pi(X), \mathscr{F}(d)) \longrightarrow \mathscr{F}(d)|_{P^{(1)}}$$

followed by k. As explained in the proof of [EW15, Lemma 5.2(a)], this restriction is surjective for $d \gg 0$ by Serre vanishing. Thus the asymptotics for $R_{n,d}$ and $\mathscr{F}(d)|_{P^{(1)}}$ at P are the same, and we may use the latter to compute probabilities.

Define $\mathcal{T}_{d,P}$ to be the preimage of $\bigcap_{x\in\pi^{-1}(P)}(\ker\varphi_x)^c$ under k. This Taylor condition captures the property that a section is smooth at all points of $\pi^{-1}(P)$.

Let $c(d) = \frac{d}{\max\{M+1,p\}}$ and define the low, medium, and high degree points as usual. Then conditions (a)-(c) of Proposition 3.1.3 are exactly [EW15, Lemmas 3.2-3.4].

Remark 3.2.3. Combining the methods in Sections 3.2.2 and 3.2.3, one can do the same analysis for the semiample complete intersection result of [Gru22, Chapter 2].

3.2.4 Containing a closed subscheme

Furthering the comparison with classical Bertini theorem, Poonen showed in [Poo08] that, given some dimension assumptions, the hypersurfaces H_f that intersect X smoothly can be assumed to contain a given closed subscheme Z of $\mathbb{P}^n$ so long as $Z \cap X$ is smooth. In [Wut14], Wutz removed the assumption that $Z \cap X$ be smooth. This material is also contained in the shorter preprint [Wut16]; independently, the main theorem was also proved by Gunther in [Gun17].

For Z a closed subscheme of $\mathbb{P}^n$, let $I_{Z,d}$ be the set of $f \in S_d$ that vanish on Z. Let $\mathscr{I}_Z$ be the ideal sheaf of Z and identify $I_{Z,d}$ with the global sections $H^0(\mathbb{P}^n, \mathscr{I}_Z(d))$ (cf. [GW20, Remark 13.26]).

For X a scheme locally of finite type over a field k, write X_j for the locally closed subscheme of X of points with embedding dimension j, i.e. where $\dim_{\kappa(x)}(\Omega^1_{X/k}|_x) = j$.

Theorem 3.2.4 ([Wut14, Theorem 2.1])**.** *Let X be a quasiprojective subscheme of $\mathbb{P}^n$ of dimension $m \geq 0$ over $\mathbb{F}_q$. Let Z be a closed subscheme of $\mathbb{P}^n$ and let $V := Z \cap X$. Define*

$$\mathcal{P}_d := \{\, f \in I_{Z,d} \mid H_f \cap X \textit{ is smooth of dimension } m-1 \,\}.$$

(a) *If* $\max_{0\le j\le m-1}\{j+\dim V_j\} < m$ *and* $V_m = \emptyset$*, then*

$$\lim_{d\to\infty} \mathrm{Prob}(f \in \mathcal{P}_d) = \frac{\zeta_V(m+1)}{\zeta_X(m+1)\prod_{j=0}^{m-1}\zeta_{V_j}(m-j)} = \frac{1}{\zeta_{X-V}(m+1)\prod_{j=0}^{m-1}\zeta_{V_j}(m-j)}.$$

(b) *If* $\max_{0\le j\le m-1}\{j+\dim V_j\} \ge m$ *or* $V_m \ne \emptyset$*, then* $\lim_{d\to\infty}\mathrm{Prob}(f\in\mathcal{P}_d) = 0$.

In the context of Proposition 3.1.3, set $Y = \mathbb{P}^n_{\mathbb{F}_q}$ and $\mathscr{F} = \mathscr{I}_Z$. For a section $f \in I_{Z,d}$, $H_f \cap X$ is smooth of dimension $m-1$ at a closed point $x \in X$ if and only if f does not vanish under the map

$$\phi_d : I_{Z,d} = H^0(\mathbb{P}^n, \mathscr{I}_Z(d)) \longrightarrow H^0(x^{(1)}, \mathscr{I}_Z \cdot \mathscr{O}_{x^{(1)}})$$

by [Wut14, Lemma 2.9]. This map is is surjective for $d \gg 0$ ([Wut14, Lemma 2.8]) and factors through the map $I_{Z,d} \to \mathscr{I}_Z(d)_x/\mathfrak{m}_x^2$.

For each $d \ge 0$ and closed $x \in \mathbb{P}^n$, define

$$\mathcal{T}_{d,x} := \begin{cases} \left\{ g_x \in \mathscr{I}_Z(d)_x/\mathfrak{m}_x^2 \;\middle|\; \text{image of } g_x \text{ in } \mathscr{I}_Z\cdot\mathscr{O}_{X,x}/\mathfrak{m}_x^2 \text{ is nonzero} \right\} & x \in X \\ \mathscr{I}_Z(d)_x/\mathfrak{m}_x^2 & x \notin X \end{cases}$$

This Taylor condition captures the property that $H_f \cap X$ is smooth of dimension $m-1$ and H_f contains Z. Fix an integer c such that $S_1 I_{Z,d} = I_{Z,d+1}$ for all $d \ge c$ (see [Wut14, remarks afer Lemma 2.7]). Set $c(d) = \frac{d-c}{m+1}$. Then in the setting of part (a) of Theorem 3.2.4, conditions (a)-(c) of Proposition 3.1.3 are exactly [Wut14, Lemmas 2.12, 2.15, 2.16, and 2.19].

In the setting of (b) where $\max_{0\le j\le m-1}\{j+\dim V_j\} \ge m$ or $V_m \ne \emptyset$, the same computations above apply. However, in the first case, Wutz shows the partial product resulting from low degree points is bounded by the inverse of the partial Euler product of a zeta function evaluated at one of its poles; thus, as $e \to \infty$ and more terms of the product are included, the product goes to zero. In the second case, the inverse of the zeta function for V_m will contain a term $1 - q^{-(m-m)\deg(x)}$, hence the entire product is zero.

Remark 3.2.5. This shows that it *does not* follow from the setup of Proposition 3.1.3 that the resulting product is zero if and only if one of the terms is zero. Note, though, that this is the case for all other theorems discussed in this section.

Remark 3.2.6. Capturing the related but slightly different results of [Wut17, Theorems 2.1 and 3.1] under a modified version of Proposition 3.1.3 would be more difficult since there Wutz is prescribing an additional condition on the dimension of the singular locus $(H_f \cap X)_{\mathrm{sing}}$, which is not a Taylor condition.

3.2.5 Smooth-agnostic Taylor conditions

We restate Theorem A from Chapter 2.

Theorem A. *Let X be a quasiprojective subscheme of $\mathbb{P}^n_{\mathbb{F}_q}$ of dimension m with locally closed embedding ι. Let $\mathcal{Q}$ be a locally free quotient of $\iota^*\Omega^1_{\mathbb{P}^n}$ of rank $\ell \geq m$, and let $\mathcal{K}$ denote the kernel of $\iota^*\Omega^1_{\mathbb{P}^n} \twoheadrightarrow \mathcal{Q}$. For each d, define*

$$\mathcal{E}_d := \left(\iota^*\mathcal{P}^1(\mathcal{O}_{\mathbb{P}^n}(d))\right) / \mathcal{K}(d)$$

where we view $\mathcal{K}(d)$ as a subsheaf of $\iota^*\mathcal{P}^1(\mathcal{O}_{\mathbb{P}^n}(d))$ via the exact sequence

$$0 \longrightarrow \iota^*\Omega^1_{\mathbb{P}^n}(d) \longrightarrow \iota^*\mathcal{P}^1(\mathcal{O}_{\mathbb{P}^n}(d)) \longrightarrow \mathcal{O}_X(d) \longrightarrow 0.$$

This defines a 1-infinitesimal Taylor condition $\mathcal{T}_d$ on $\mathbb{P}^n$ such that at each closed point x, $\mathcal{T}_{d,x} \subseteq \mathcal{O}_{\mathbb{P}^n}(d)_x/\mathfrak{m}_x^2$ is given by not vanishing in the fiber of $\mathcal{E}_d$ at x. By convention, $\mathcal{T}_d$ is always satisfied if $x \notin X$.

Define

$$\mathcal{P}_d := \left\{ f \in S_d \mid f \text{ satisfies } \mathcal{T}_d \text{ at all closed } x \in \mathbb{P}^n \right\}.$$

Then

$$\lim_{d\to\infty} \operatorname{Prob}(f \in \mathcal{P}_d) = \prod_{\text{closed } x \in X} \left(1 - q^{-(\ell+1)\deg(x)}\right) = \zeta_X(\ell+1)^{-1}.$$

In the context of Proposition 3.1.3, set $Y = \mathbb{P}^n$ and $\mathcal{F} = \mathcal{O}_{\mathbb{P}^n}$ on $\mathbb{P}^n$. Then conditions (a)-(c) are exactly Lemmas 2.4.1 to 2.4.3.

3.2.6 Related results in the literature

Other results that fall under the framework of Proposition 3.1.3 include [Lin17, Theorem 1.1] and [GK23, Proposition 4.8]. Details are omitted since the steps are nearly identical to the examples above.

While not strictly related to Taylor conditions, the results in [Ngu05], [Poo13], [CP16], [BE19], and [GK23, §4 and §5] are worth consideration for future attempts at axiomatization of Bertini-type theorems over finite fields.

Chapter 4

Equidistribution and arithmetic Λ-distributions

The following chapter is joint work by the author and Sean Howe.

4.1 Introduction

In [How24], Sean Howe introduced a theory of probability in λ-rings in order to provide a concise language for describing random variables valued in multisets of complex numbers. The main application was a comparison between zero distributions for certain families of function field L-functions and associated eigenvalue distributions in random matrix statistics ([How24, Theorems B and C]), and the key arithmetic input for the two types of families treated in [How24] was Poonen's sieve for hypersurface sections ([Poo04]). The purpose of the present work is to generalize and abstract the method used in [How24] to compute arithmetic Λ-distributions or, equivalently, their associated σ-moment-generating functions, and then to apply this generalization to compute arithmetic Λ-distributions in new cases.

To that end, we first formulate in Definition 4.4.1 a general notion of equidistribution for a sequence of families of λ-probability spaces parameterized by an admissible $\mathbb{Z}$-set (an abstraction of the $\overline{\mathbb{F}}_q$-points of an algebraic variety over $\mathbb{F}_q$). In Theorem 4.4.3 we show that, if equidistribution holds, then for certain sequences of random variables constructed by integrating over these families, the associated sequence of σ-moment-generating functions converges to an explicit motivic Euler product. We apply this abstract result to compute asymptotic Λ-distributions for L-function and zeta function statistics in more settings where Poonen's sieve has been generalized (using the original sieve, one recovers [How24, Theorems B and C]).

In particular, in Theorem D we combine our systematization with the generalization of

Poonen's sieve given in [BK12] to compute the asymptotic Λ-distributions of the zeroes of the L-functions of vanishing cohomology of smooth complete intersections and compare these with the associated random matrix Λ-distributions. This generalizes the case of smooth hypersurface sections treated in [How24, Theorem C]. We also treat natural families arising from the semiample version of Poonen's sieve of [EW15] and the "smooth-agnostic" generalization of Poonen's sieve developed in Chapter 2.

The key new tool that allows us to systematize and abstract the arguments of [How24] is a motivic Euler product adapted to point-counting. We continue the introduction by explaining how this notion arises naturally in the problems we consider.

4.1.1 Equidistribution, independence, and σ-moment-generating functions

The arithmetic random variables studied in [How24] are of the following form: Fix a finite field κ and algebraic closure $\overline{\kappa}$. For each $d \geq 1$, one defines a λ-probability space where the random variables are functions on the set U_d of the degree d homogeneous polynomials in $n+1$ variables with coefficients in $\overline{\kappa}$ satisfying a transversality condition with respect to a fixed smooth subscheme of $\mathbb{P}^n_\kappa$. For each $P \in \mathbb{P}^n(\overline{\kappa})$, one defines a random variable $X_{d,P}$ on U_d whose value on a homogeneous polynomial F depends only on the Taylor expansion of F at P. One then obtains a new random variable X_d on U_d by "summing" the random variables $X_{d,P}$ over all points $P \in \mathbb{P}^n(\overline{\kappa})$, and one would like to understand the asymptotic distribution of X_d as $d \to \infty$ — there is hope of this because Poonen's sieve ([Poo04]) implies the Taylor expansions equidistribute in a certain natural sense.

In this theory, the random variables are valued in $W(\mathbb{C})$, the ring of big Witt vectors of $\mathbb{C}$, and the right notion of a distribution is a Λ-distribution: when the random variable is valued in $\mathbb{Z}_{\geq 0}[\mathbb{C}] \subseteq W(\mathbb{C})$, i.e. in multisets of complex numbers, the Λ-distribution encodes the averages of all symmetric functions of the multiset, and convergence in $W(\mathbb{C})$ is simply convergence of all of these averages.

One of the key ideas in [How24] is to encode the Λ-distribution using the σ-moment-generating function, defined for a $W(\mathbb{C})$-valued random variable X as

$$\mathbb{E}[\mathrm{Exp}_\sigma(Xh_1)] \in \Lambda^\wedge_{W(\mathbb{C})}$$

where Exp_σ is the plethystic exponential, $h_1 = t_1 + t_2 + t_3 + \ldots$, and $\Lambda^\wedge_{W(\mathbb{C})}$ is the ring of symmetric power series with coefficients in $W(\mathbb{C})$. These behave much like the usual moment-generating functions in classical probability theory: in particular, in the above setting, we expect that, as

$d \to \infty$, the random variables $X_{d,P}$ will all be independent so that their moment-generating functions should multiply to give

$$\lim_{d\to\infty} \mathbb{E}[\mathrm{Exp}_\sigma(X_d h_1)] = \prod_{P\in\mathbb{P}^n(\overline{\kappa})} \lim_{d\to\infty} \mathbb{E}[\mathrm{Exp}_\sigma(X_{d,P} h_1)].$$

Moreover, we expect that each of the individual random variables $X_{d,P}$ should approach the distribution of a random variable $\mathcal{X}_P$ defined independently of d on the space of Taylor expansions at P, so that this should simplify further to

$$\prod_{P\in\mathbb{P}^n(\overline{\kappa})} \mathbb{E}[\mathrm{Exp}_\sigma(\mathcal{X}_P h_1)].$$

The main difficulty in making this heuristic precise is that it is completely unclear how one should actually define the product over $\mathbb{P}^n(\overline{\mathbb{F}}_q)$. In [How24], we made an ad hoc argument to get around this, exploiting that, in the cases treated there, the Λ-distribution of each $\mathcal{X}_P$ is the same. Under this constraint, one finds a natural candidate for the infinite product by using the pre-λ power structure (see [How24, §2.6]).

In the present work, we address the problem head-on by using the plethystic exponential and logarithm to define motivic Euler products for admissible $\mathbb{Z}$-sets — see Definition 4.3.1. This is motivated by joint work of Sean Howe with Margaret Bilu and Ronno Das ([BDH25]) where it is shown that, when working with Grothendieck rings of varieties, the same formula recovers the motivic Euler products of Bilu ([Bil23]). Even when the infinite product can be expressed in the pre-λ power structure as in [How24], the perspective adopted here gives clearer proofs than the previous ad hoc method.

Our motivic Euler products can also be treated from the perspective of Eulerian formalisms developed in [BDH25], but here we give a more direct and independent treatment. A key point that is specific to the case of admissible $\mathbb{Z}$-sets is Proposition 4.3.7, which explains how to compute the motivic Euler products in terms of classical Euler products. This formula is what allows us to build a bridge between the abstract formulation of equidistribution (Definition 4.4.1), which is adapted to comparison with Poonen's sieve, and the computation of σ-moment-generating functions that carries out the heuristic described above (Theorem 4.4.3).

4.1.2 Applications

Combining Theorem 4.4.3 with generalizations of Poonen's sieve, we can compute asymptotic distributions of many zeta function and L-function random variables. As an illustration, we state now a generalization of [How24, Theorem C] to complete intersections (that uses the generalization of Poonen's sieve in [BK12] to obtain the equidistribution result needed to apply Theorem 4.4.3). Afterwards we will briefly discuss our other applications.

4.1.2.1 Fix a finite field κ and an algebraic closure $\overline{\kappa}$. For $n \geq 0$, $r \geq 1$, suppose $Y \subseteq \mathbb{P}^n_\kappa$ is a smooth closed geometrically connected subscheme of dimension $m+r$. For $\underline{d} = (d_1, \ldots, d_r) \in \mathbb{N}^r$, let $U_{\underline{d}}$ be the set of tuples $\underline{F} = (F_1, \ldots, F_r)$ of homogeneous polynomials of degrees $d_1, \ldots, d_r$ in $n+1$ variables with coefficients in $\overline{\kappa}$ such that the subschemes $Y, V(F_1), \ldots, V(F_r)$ are transverse at any point of intersection (i.e., the intersection of their tangent spaces at such a point is m-dimensional). Then the scheme-theoretic intersection $C_{\underline{F}} = Y \cap V(F_1) \cap \ldots \cap V(F_r)$ is a smooth complete intersection in $Y_{\overline{\kappa}}$ of dimension m. If we write $\kappa(\underline{F})$ for the subfield of $\overline{\kappa}$ generated by κ and the coefficients of $F_1, \ldots, F_r$, then $C_{\underline{F}}$ is naturally defined over $\kappa(\underline{F})$, and as such admits a Hasse-Weil zeta function

$$Z_{C_{\underline{F}}}(t) = \prod_{y \in |C_{\underline{F}}|} \frac{1}{1 - t^{\deg y}}.$$

Here $|C_{\underline{F}}|$ denotes the closed points of $C_{\underline{F}}/\kappa(\underline{F})$. It follows from the Grothendieck-Lefschetz trace formula and the strong Lefschetz theorem in étale cohomology that

$$Z_{C_{\underline{F}}}(t) = \mathcal{L}_{C_{\underline{F}}}(t)^{(-1)^m} Z_0(t)$$

where $Z_0(t)$ depends only on $Y_{\kappa(\underline{F})}$ and $\mathcal{L}_{C_{\underline{F}}}(t)$ is the characteristic power series of geometric Frobenius acting on the vanishing cohomology of $C_{\underline{F}}$ (that is, the part of the cohomology which does not "come from" $Y_{\kappa(\underline{F})}$). For $q_{\underline{F}} := \#\kappa(\underline{F})$, the reciprocal poles of $\mathcal{L}_{C_{\underline{F}}}(t)$ are $q_{\underline{F}}$-Weil numbers of weight m, i.e., they are algebraic integers whose conjugates all have absolute value $q_{\underline{F}}^{\frac{m}{2}}$. One expects the renormalized characteristic series

$$\mathcal{L}_{C_{\underline{F}}}(tq_{\underline{F}}^{\frac{-m}{2}})$$

to behave, on average, like the characteristic power series of a random matrix in an orthogonal group if $m > 0$ is even, a compact symplectic group if m is odd, or the symmetric group in its standard irreducible representation if $m = 0$.

4.1.2.2 We compute the asymptotic Λ-distribution of the random variable $X_{\underline{d}}$ on $U_{\underline{d}}$ given by

$$X_{\underline{d}}(\underline{F}) = \mathcal{L}_{C_{\underline{F}}}(tq_{\underline{F}}^{\frac{-m}{2}}) \in 1 + t\mathbb{C}[[t]] = W(\mathbb{C})$$

(equivalently, sending F to the multiset of reciprocal poles as an element of $\mathbb{Z}_{\geq 0}[\mathbb{C}] \subseteq W(\mathbb{C})$, i.e. to the eigenvalues of the associated matrix) and compare it to one of the classical group random matrix Λ-distributions computed in [How24, Theorem A].

To state the result, we introduce some notation: We write $[Y(\overline{\kappa})] = Z_Y(t) \in W(\mathbb{C})$ and $[H^i(Y)] \in 1 + t\mathbb{C}[[t]] = W(\mathbb{C})$ for the characteristic power series of the geometric Frobenius acting on the i^{th} étale cohomology group of $Y_{\overline{\kappa}}$ (which are relevant because the persistent factor $Z_0(t)$ of $Z_{C_{\underline{F}}}(t)$ can be expressed in terms of $[H^i(Y)]$, $i \le m$). For $z \in \mathbb{C}$, we write $[z]$ for the element $\frac{1}{1-tz}$ in $1 + t\mathbb{C}[[t]] = W(\mathbb{C})$.

We write e_i for the i^{th} elementary symmetric polynomial and h_i for the i^{th} complete symmetric polynomial (see [How24, §2.1] for the general notation and results on symmetric functions that we use).

We set $q := \#\kappa$. We write $L(a,b,c) = \prod_{j=0}^{c-1}(1-a^{-(b-j)})$; note that $L(q, m+r, r)$ is the probability that r vectors in $\mathbb{F}_q^{m+r}$ are linearly independent (cf. [BK12, p.2]).

We write $\lim_{\underline{d} \xrightarrow{*} \infty}$ for a limit where each $d_i \to \infty$ and $\max(d_i)^{m+r} q^{\frac{-\min(d_i)}{m+r+1}} \to 0$.

Finally, we write $W(\mathbb{C})^{\text{bdd}}$ for the subring of $W(\mathbb{C})$ consisting of elements with bounded ghost components (which, as in [How24, §9], encodes some big O notation).

When $r = 1$, the following result is [How24, Theorem C]; we refer the reader to loc. cit. and the surrounding discussion for more on its classical interpretations.

Theorem D. *With notation as above,*

$$\lim_{\underline{d} \xrightarrow{*} \infty} \mathbb{E}[\mathrm{Exp}_\sigma(X_{\underline{d}} h_1)] = \left(1 + p \sum_{i \ge 1} [q^{-im/2}] \epsilon^i f_i\right)^{[Y(\overline{\kappa})]} \cdot \mathrm{Exp}_\sigma(\mu h_1) \tag{4.1.1}$$

where $f_i = e_i$ and $\epsilon = -1$ if m is odd and $f_i = h_i$ and $\epsilon = 1$ if m is even,

$$p = \frac{[q]^{-r} L([q], m+r, r)}{1 - [q]^{-r} + [q]^{-r} L([q], m+r, r)}, \quad \textit{and}$$

$$\mu = -\epsilon \left(\sum_{i=0}^{m-1} (-1)^i \left([q^{\frac{-m}{2}}] + [q^{\frac{m-2i}{2}}] \right) [H^i(Y)] \right) - [q^{-m/2}][H^m(Y)].$$

In particular, modulo $[q^{-\frac{1}{2}}]\Lambda^\wedge_{W(\mathbb{C})^{\text{bdd}}}$ this agrees with:

(a) *For $n > 0$ even,* $\mathrm{Exp}_\sigma(h_2)$ *(the asymptotic σ-moment-generating function for orthogonal random matrices ([How24, Theorem A-(1)])).*

(b) *For n odd,* $\mathrm{Exp}_\sigma(e_2)$ *(the asymptotic σ-moment-generating function for symplectic random matrices ([How24, Theorem A-(2)])).*

(c) *For $n = 0$,* $\mathrm{Exp}_\sigma(h_2 + h_3 + \ldots)$ *(the asymptotic σ-moment-generating function for the standard irreducible representations of symmetric groups ([How24, Example 1.2.1])).*

To obtain the limiting σ-moment-generating function in Theorem D, as in the proof of [How24, Theorem C], we use formal properties of independence to reduce to a geometric version computing the distribution of the random variable $X_{\underline{d}}$ sending $\underline{F}$ to $Z_{C_{\underline{F}}}(t)$. This geometric result is given in Theorem 4.6.2, and generalizes [How24, Theorem 8.3.1] (we use [How25, Theorem 2.2.1] to handle a negative sign when m is odd instead of carrying out the computation separately as in [How24]). The comparisons with random matrix statistics are then deduced as in [How24, Proposition 9.2.2].

4.1.2.3 In Theorem 4.5.4 we give a different generalization of [How24, Theorem 8.3.1], treating zeta functions of hypersurface sections with more exotic transversality conditions — this is deduced from Theorem 4.4.3 using the smooth-agnostic extension of Poonen's sieve in Chapter 2. In Section 4.5.2 we also explain how the computations in [How24, Theorem B], which gave the asymptotic Λ-distributions of L-functions of certain families of Dirichlet characters, can be recovered from the perspective adopted here (this requires only Poonen's original sieve as input into Theorem 4.4.3).

In Theorem 4.7.2 we compute the asymptotic Λ-distributions of the zeta functions of degree $(2, d)$ curves on Hirzebruch surfaces — this computation is deduced from Theorem 4.4.3 using the semiample extension of Poonen's sieve in [EW15], and it refines [EW15, Theorem 9.9-(b)] (see the start of Section 4.7.2 for further discussion).

Remark 4.1.1. Analogous results in the Grothendieck rings of varieties and Hodge structures will be given in [BH25]. In that setting, the results and methods of [BH21] provide a uniform treatment of the input that is analogous to that coming from Poonen's sieve in the point-counting setting. We recall that results giving asymptotic point-counts or traces of Frobenius and results giving asymptotics in the Grothendieck ring of varieties with respect to the dimension filtration do not imply one another, and typically require different methods (see, e.g., [BDH22, §1] for a detailed discussion).

4.1.3 Organization

In Section 4.2 we set up some basic results on (pre-)λ-rings, admissible $\mathbb{Z}$-sets and their $W(\mathbb{C})$-valued functions, and λ-probability spaces. Much of the material is recalled from [How24], but we also give a few new results and definitions adapted to handling families of λ-probability spaces parameterized by admissible $\mathbb{Z}$-sets. In Section 4.3 we define motivic Euler products with respect to a map of admissible $\mathbb{Z}$-sets and establish the basic properties of this operation.

After these preliminaries, in Section 4.4 we define our notion of equidistribution and establish the abstract form of our main result, Theorem 4.4.3. In Section 4.5, Section 4.6, and Section 4.7, respectively, we then use generalizations of Poonen's sieve to establish equidistribution and obtain applications of Theorem 4.4.3 for homogeneous polynomials (using the generalization of Poonen's sieve of Chapter 2), tuples of homogeneous polynomials (using the generalization of Poonen's sieve in [BK12]), and sections of semiample line bundles (using the generalization of Poonen's sieve in [EW15]), respectively. In particular, Theorem D is established in Section 4.6.

4.1.4 Acknowledgments

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4.2 Preliminaries

In this section we discuss some basic notions on (pre-)λ-rings, admissible $\mathbb{Z}$-sets and their $W(\mathbb{C})$-valued functions, and λ-probability. Except for some new base change properties that will be useful for working with geometric families of random variables, this material is treated in more detail in [How24, §2, 3, and 5]. We give citations to [How24]; citations to earlier work on some of these topics can be found in [How24].

4.2.1 pre-λ-rings

4.2.1.1 We will use the notation for symmetric functions described in [How24, §2.1]. In particular, we write Λ for the ring of symmetric functions, and h_i (resp. e_i, resp. p_i) denotes the i^{th} complete (resp. elementary, resp. power sum) symmetric function. For $\tau = (\tau_1, \tau_2, \ldots)$ a partition, $h_\tau = h_{\tau_1} h_{\tau_2} \cdots$, and similarly for p_τ and e_τ. The monomial symmetric function m_τ is the formal sum of all distinct permutations of the monomial $t_1^{\tau_1} t_2^{\tau_2} \cdots$. We will frequently use that

$$h_1 = e_1 = p_1 = m_{(1,0,0,\ldots)} = t_1 + t_2 + t_3 + \ldots.$$

4.2.1.2 Recall from [How24, §2.2 and §2.4] that a (pre-)λ-ring R is a ring equipped with a plethystic action of the ring Λ of symmetric functions, written $a \circ r$ for $a \in \Lambda$ and $r \in R$, satisfying certain natural compatibilities.

4.2.1.3 We let $W(\mathbb{C}) = \mathrm{Hom}_{\mathrm{ring}}(\Lambda, \mathbb{C})$ denote the ring of big Witt vectors of $\mathbb{C}$. We refer the reader to [How24, §5.1] for an overview of its properties; here we briefly recall the structures we will use. As an additive group, $W(\mathbb{C})$ is naturally isomorphic to $1 + t\mathbb{C}[[t]]$ under multiplication by

$$w \mapsto 1 + w(h_1)t + w(h_2)t^2 + \dots.$$

For $w \in W(\mathbb{C})$, we write $w(t^i)$ for the element obtained by substituting t^i for t in this presentation. As a ring, $W(\mathbb{C})$ is naturally isomorphic to $\prod_{k\geq 1}\mathbb{C}$ by

$$w \mapsto (w(p_1), w(p_2), \dots).$$

For $w \in W(\mathbb{C})$, and $k \geq 1$ we write $w_k = w(p_k)$ for its k^{th} component in this product presentation, called the k^{th} ghost component. There is a λ-ring structure on $W(\mathbb{C})$ determined by the Adams operations

$$p_i \circ (w_1, w_2, \dots) = (w_i, w_{2i}, \dots) \tag{4.2.1}$$

and we have

$$w(t^i)_k = i\,w_{i/k} \tag{4.2.2}$$

where we take $w_{i/k}$ to be zero if i/k is not a positive integer.

4.2.1.4 Recall from [How24, Section 2.3] that, for any (pre-)λ-ring R, we have a natural (pre-)λ-ring structure on

$$R[[\underline{t}_{\mathbb{N}}]] = \varprojlim_n R[[t_1, \dots, t_n]]$$

extending the (pre-)λ-ring structure on R and such that $p_i \circ t_j = t_j^i$. It is moreover a filtered (pre-)λ-ring for the filtration by monomial degree. In particular, as in [How24, §2.5], we have the σ-exponential

$$\begin{aligned} \mathrm{Exp}_\sigma : \mathrm{Fil}^1 R[[\underline{t}_{\mathbb{N}}]] &\longrightarrow 1 + \mathrm{Fil}^1 R[[\underline{t}_{\mathbb{N}}]] \\ F &\longmapsto \textstyle\sum_{k\geq 0} h_k \circ F \end{aligned}$$

and its inverse (which exists for formal reasons)

$$\mathrm{Log}_\sigma : 1 + \mathrm{Fil}^1 R[[\underline{t}_{\mathbb{N}}]] \longrightarrow \mathrm{Fil}^1 R[[\underline{t}_{\mathbb{N}}]].$$

All of these constructions can be restricted to the (pre-)λ-subring $\Lambda_R^\wedge \subseteq R[[\underline{t}_{\mathbb{N}}]]$ of symmetric power series.

4.2.1.5 Recall that, for $F \in 1 + \mathrm{Fil}^1 R[[\underline{t}_{\mathbb{N}}]]$ and $N \in R[[\underline{t}_{\mathbb{N}}]]$ we have an associated pre-λ power as in [How24, Definition 2.6.1],

$$F^N := \mathrm{Exp}_\sigma(N \cdot \mathrm{Log}_\sigma(F)).$$

4.2.1.6 At certain points, we will also wish to use the coefficient-wise pre-λ-ring structure on $R[[\underline{t}_{\mathbb{N}}]]$ or $\Lambda_R^\wedge$, which we denote by $*$, i.e. $f * \sum r_{\underline{i}} \underline{t}^{\underline{i}} = \sum (f \circ r_{\underline{i}}) \underline{t}^{\underline{i}}$.

4.2.2 Admissible $\mathbb{Z}$-sets

4.2.2.1 Recall from [How24, §5.2] that an admissible $\mathbb{Z}$-set is a set V with an action of $\mathbb{Z}$ such that $V = \cup_{k \geq 1} V^{k\mathbb{Z}}$ and, for any $k \geq 1$, $V^{k\mathbb{Z}}$ is finite. We write $|V|$ for the set of $\mathbb{Z}$-orbits in V and, for $v \in V$, we write $|v|$ for the orbit containing v. The degree of a point or orbit is the size of the orbit; we write this as $\deg(v)$ or $\deg(|v|)$. We write $\mathbf{k} = \mathbb{Z}/k\mathbb{Z}$ (as an admissible $\mathbb{Z}$-set). For any admissible $\mathbb{Z}$-set V, we write

$$[V] = \prod_{|v| \in |V|} \frac{1}{1 - t^{\deg(v)}} \in 1 + t\mathbb{C}[[t]] = W(\mathbb{C}).$$

In ghost coordinates, we have $[V] = (\#V(\mathbf{1}), \#V(\mathbf{2}), \ldots)$ (see [How24, Lemma 5.2.3]).

4.2.2.2 Recall from [How24, Definition 5.2.5] that, if $V \to B$ is a map of admissible $\mathbb{Z}$-sets, then, for any $b \in B$, we equip the fiber V_b with the structure of an admissible $\mathbb{Z}$-set by multiplying the action on V by $\deg(b)$ so that it restricts to an action on V_b.

4.2.2.3 For V an admissible $\mathbb{Z}$-set and k a positive integer, we write $V_{\mathbf{k}}$ for the same set but with the action of $\mathbb{Z}$ multiplied by k. If $\varphi : V \to B$ is a map of admissible $\mathbb{Z}$-sets then we write $\varphi_{\mathbf{k}}$ for the induced map $V_{\mathbf{k}} \to B_{\mathbf{k}}$.

We can identify $V_{\mathbf{k}}$ with the fiber of the projection map $V \times \mathbf{k} \to \mathbf{k}$ over $1 \in \mathbf{k}$.

4.2.2.4 Admissible $\mathbb{Z}$-sets give an abstract formalism for studying the points of algebraic varieties over finite fields: for $\mathbb{F}_q$ a finite field of cardinality q, fix an algebraic closure $\overline{\mathbb{F}}_q$ and write $\mathbb{F}_{q^k}$ for the unique subfield of cardinality q^k. Then, the set of $\overline{\mathbb{F}}_q$-points $Y(\overline{\mathbb{F}}_q)$ of an algebraic variety over $Y/\mathbb{F}_q$ is an admissible $\mathbb{Z}$-set with 1 acting as the geometric Frobenius; in this case, $|Y(\overline{\mathbb{F}}_q)|$ can be identified with the closed points of Y, and $[Y(\overline{\mathbb{F}}_q)]$ is an incarnation of the zeta function of Y (see also [How24, Example 5.2.2]). The set $Y(\overline{\mathbb{F}}_q)_{\mathbf{k}}$ is then $Y_{\mathbb{F}_{q^k}}(\overline{\mathbb{F}}_q)$, i.e. the set $Y(\overline{\mathbb{F}}_q)$ but with $1 \in \mathbb{Z}$ acting by the q^k-power geometric Frobenius instead of the q-power geometric Frobenius.

4.2.3 $W(\mathbb{C})$-valued functions

4.2.3.1 For V an admissible $\mathbb{Z}$-set, we write $C(V, W(\mathbb{C}))$ for the set of functions from V to $W(\mathbb{C})$ that are constant on $\mathbb{Z}$-orbits. It is a λ-ring with the pointwise λ-ring structure, and, given $\varphi : V \to B$ a map of admissible $\mathbb{Z}$-sets, there is a natural pullback map ([How24, Definition 5.3.4-(1)])

$$\varphi^* : C(B, W(\mathbb{C})) \longrightarrow C(V, W(\mathbb{C}))$$

that is a map of λ-rings, and a natural integration-over-fibers map ([How24, Definition 5.3.4-(2)])

$$\varphi_! : C(V, W(\mathbb{C})) \longrightarrow C(B, W(\mathbb{C}))$$

that is φ^*-linear ([How24, Lemma 5.3.7]). When the map φ is clear from context, we write $\int_{V/B}$ in place of $\varphi_!$. If $B = \mathbf{1}$ is the final object, we may also write $\int_V$ in place of $\int_{V/\mathbf{1}}$. We recall the formulas for φ^* and $\int_{V/B}$, since they will be used below:

- For $g \in C(B, W(\mathbb{C}))$,
$$(\varphi^* g)(v) = p_{\frac{\deg(v)}{\deg(\varphi(v))}} \circ g(\varphi(v)).$$
- For $f \in C(V, W(\mathbb{C}))$,
$$\left(\int_{V/B} f\right)(b) = \int_{V_b} f = \sum_{|v| \in |V_b|} f(t^{\deg(v)}).$$

Example 4.2.1 (See Example 5.3.5 of [How24])**.** For V an admissible $\mathbb{Z}$-set,

$$\int_V 1 = [V]$$

where here 1 denotes the constant function on V with value the unit in $W(\mathbb{C})$ (the unit in $W(\mathbb{C})$ is the element $\frac{1}{1-t}$ in the identification $1 + t\mathbb{C}[[t]] = W(\mathbb{C})$).

We will need the following base change formula relating pullback and integration over fibers that was not made explicit in [How24]:

Lemma 4.2.2. *If*

$$\begin{array}{ccc} V_1 & \xrightarrow{\varphi} & V_2 \\ \downarrow & & \downarrow \\ B_1 & \xrightarrow{\psi} & B_2 \end{array}$$

is a cartesian diagram of admissible $\mathbb{Z}$-sets then, for $f \in C(V_2, W(\mathbb{C}))$,

$$\psi^* \int_{V_2/B_2} f = \int_{V_1/B_1} \varphi^* f.$$

Proof. We write $V := V_2$ so that $V_1 = V \times_{B_2} B_1$. Then, for $b \in B_1$ and $a := \deg(b)/\deg(\psi(b))$,

$$\begin{aligned}\left(\psi^* \int_{V/B_2} f\right)(b) &= p_a \circ \left(\int_{V/B_2} f\right)(\psi(b)) \\ &= p_a \circ \int_{V_{\psi(b)}} f|_{V_{\psi(b)}} \\ &= \sum_{|v| \in V_{\psi(b)}} p_a \circ \left(f(|v|)(t^{\deg(v)})\right), \end{aligned} \tag{4.2.3}$$

where the three equalities are immediate from the definition of pullback and integration. On the other hand, again immediately from the definitions,

$$\left(\int_{V \times_{B_2} B_1/B_1} \varphi^* f\right)(b) = \sum_{|w| \in (V \times_{B_2} B_1)_b} \left(p_{\deg(w)/\deg(\varphi(w))} \circ f(|w|)\right)(t^{\deg(w)}). \tag{4.2.4}$$

Now, $(V \times_{B_2} B_1)_b$ is identified with $V_{\psi(b)}$, but with the action multiplied by a. Thus each orbit $|v|$ in $V_{\psi(b)}$, viewed as a subset of $(V \times_{B_2} B_1)_b$, decomposes into $\mu_{|v|} := \gcd(\deg(v), a)$ orbits of size $\deg(v)/\mu_{|v|}$, whose points w satisfy $\deg(w)/\deg(v) = a/\mu_{|v|}$. Thus, we can rewrite the sum on the right of Equation (4.2.4) as

$$\sum_{|v| \in V_{\psi(b)}} \mu_{|v|} \left(p_{a/\mu_{|v|}} \circ f(|v|)\right)(t^{\deg(v)/\mu_{|v|}}). \tag{4.2.5}$$

To see that Equation (4.2.5) agrees with the right of Equation (4.2.3), we use Equation (4.2.1) and Equation (4.2.2) to compute ghost coordinates of the terms appearing as

$$\left(p_a \circ \left(f(|v|)(t^{\deg(v)})\right)\right)_i = \deg(v) f(|v|)_{ai/\deg(v)}$$

and

$$\left(\left(p_{a/\mu_{|v|}} \circ f(|v|)\right)(t^{\deg(v)/\mu_{|v|}})\right)_i = \frac{\deg(v)}{\mu_{|v|}} f(|v|)_{ai/\deg(v)}.$$

Since a multiple of $\mu_{|v|}$ appears in Equation (4.2.5), we conclude. □

4.2.3.2 Recall from 4.2.2.3 that, for V an admissible $\mathbb{Z}$-set and k a positive integer, we have defined $V_{\mathbf{k}}$ by multiplying the $\mathbb{Z}$-action by k.

If $f \in C(V, W(\mathbb{C}))$ and $k \geq 1$ then from f we obtain $f_{V_{\mathbf{k}}} \in C(V_{\mathbf{k}}, W(\mathbb{C}))$ by identifying $V_{\mathbf{k}} = (V \times \mathbf{k})_1$ as in 4.2.2.3, and pulling back first to $V \times \mathbf{k}$ then restricting. Concretely, if x is a point of degree d in V then it is a point of degree $d/\gcd(d,k)$ in $V_{\mathbf{k}}$, and $f_{V_{\mathbf{k}}}(x) = p_{k/\gcd(d,k)} \circ f(x)$. We will often write f in place of $f_{V_{\mathbf{k}}}$ when the domain is clear; we emphasize that this is not the same as transporting f naively along the identification of the underlying sets of V and V_k.

4.2.4 Some λ-probability spaces

4.2.4.1 We recall from [How24, Definition 3.1.1] that a (pre-)λ-probability space is a (pre-)λ-ring R equipped with a $\mathbb{Z}$-linear expectation functional $\mathbb{E} : R \to C$ to another ring C such that $\mathbb{E}[1_R] = 1_C$.

Given a (pre-)λ-probability space $(R, \mathbb{E})$, we refer to the elements of R as random variables. Given a random variable $X \in R$, we recall from [How24, Lemma 3.2.2] that the Λ-distribution of X as in [How24, Definition 3.1.2] is determined by the σ-moment-generating function of X ([How24, Definition 3.2.1]). The latter is defined as

$$\mathbb{E}[\mathrm{Exp}_\sigma(Xh_1)] \in \Lambda_C^\wedge$$

where $h_1 = p_1 = e_1 = t_1 + t_2 + t_3 + \ldots$ is the first complete, power sum, and elementary symmetric function, $\mathrm{Exp}_\sigma(Xh_1)$ is computed in $\Lambda_R^\wedge$ (or equivalently in $R[[\underline{t}_\mathbb{N}]]$) and the expectation is applied coefficient-wise to produce an element of $\Lambda_C^\wedge$.

4.2.4.2 Suppose V is an admissible $\mathbb{Z}$-set and $V(\mathbf{1}) \neq \emptyset$. Then, $V(\mathbf{k}) \neq \emptyset$ for all $k \geq 1$, so $[V] = (\#V(\mathbf{1}), \#V(\mathbf{2}), \ldots)$ is an invertible element of $W(\mathbb{C})$. As in [How24, §5.5], we can thus consider the λ-probability space

$$(C(V, W(\mathbb{C})), \mathbb{E})$$

where the expectation functional $\mathbb{E}$ is defined by

$$\begin{aligned} \mathbb{E} : C(V, W(\mathbb{C})) &\longrightarrow W(\mathbb{C}) \\ f &\longmapsto \frac{\int_V f}{[V]}. \end{aligned}$$

4.2.4.3 Given a pre-λ-probability space with expectation $\mathbb{E}$ valued in $W(\mathbb{C})$, we write $\mathbb{E}_k$ for the $\mathbb{C}$-valued expectation obtained by projecting to the k^{th} ghost component. We recall that, for the λ-probability space $(C(V, W(\mathbb{C})), \mathbb{E})$ as above, $\mathbb{E}_k$ can be computed naturally on a classical finite probability space ([How24, Lemma 5.5.1]). We now give a modified formulation that is more convenient for our purposes.

To state it, note that we can view $V_\mathbf{k}(\mathbf{1})$ as the subset of $\mathbb{Z}$-fixed points in $V_\mathbf{k}$.

Lemma 4.2.3. *Let V be an admissible $\mathbb{Z}$-set with $V(\mathbf{1}) \neq \emptyset$. Restriction as in 4.2.3.2 from V to $V_\mathbf{k}$ followed by naive restriction from $V_\mathbf{k}$ to $V_\mathbf{k}(\mathbf{1})$ induces a map of λ-probability spaces*

$$\mathrm{res}_k : (C(V, W(\mathbb{C})), \mathbb{E}_k) \longrightarrow (C(V_\mathbf{k}(\mathbf{1}), W(\mathbb{C})), \mathbb{E}_1).$$

In particular, for any random variable $X \in C(V, W(\mathbb{C}))$,

$$\mathbb{E}_k\left[\mathrm{Exp}_\sigma(Xh_1)\right] = \mathbb{E}_1\left[\mathrm{Exp}_\sigma(\mathrm{res}_k(X)h_1)\right].$$

Proof. Noting that evaluation at $1 \in \mathbf{k}$ gives a canonical identification $V(\mathbf{k}) = V_{\mathbf{k}}(\mathbf{1})$, this is a reformulation of [How24, Lemma 5.5.1]. □

Remark 4.2.4. The map res_k is a map of $W(\mathbb{C})$-algebras when $C(V, W(\mathbb{C}))$ is equipped with the algebra structure of pullback from a point and $C(V_{\mathbf{k}}(\mathbf{1}), W(\mathbb{C}))$ is equipped with the algebra structure sending $a \in W(\mathbb{C})$ to the constant function with value $p_k \circ a$. In the statement of [How24, Lemma 5.5.1], this twist in the algebra structure is included in the notation by writing $C(V(\mathbf{k}), W(\mathbb{C}))^{(k)}$ in place of $C(V(\mathbf{k}), W(\mathbb{C}))$.

4.2.4.4 We will also need to consider "families" of λ-probability spaces: if $V \to B$ is a morphism of admissible $\mathbb{Z}$-sets admitting a section $B \to V$, then the class $[V/B] \in C(B, W(\mathbb{C}))$ of [How24, Example 5.3.5] sending $b \in V$ to $[V_b]$ (for V_b as in 4.2.2.2) is invertible and we can consider the λ-probability space

$$(C(V, W(\mathbb{C})), \mathbb{E}_{V/B})$$

where the expectation functional $\mathbb{E}_{V/B}$ is defined by

$$\begin{aligned} \mathbb{E}_{V/B} : C(V, W(\mathbb{C})) &\longrightarrow C(B, W(\mathbb{C})) \\ f &\longmapsto \tfrac{\int_{V/B} f}{[V/B]}. \end{aligned}$$

Lemma 4.2.5. *For $f \in C(V, W(\mathbb{C}))$,*

$$\left(\mathbb{E}_{V/B}[f]\right)(b) = \mathbb{E}_{V_b}[f|_{V_b}].$$

Proof. This follows from the analogous properties in the definitions of $\int_{V/B}$ ([How24, Definition 5.3.4]) and $[V/B]$ ([How24, Example 5.3.5]):

$$\left(\mathbb{E}_{V/B}[f]\right)(b) = \frac{\left(\int_{V/B} f\right)(b)}{[V/B](b)} = \frac{\int_{V_b} f|_{V_b}}{[V_b]} = \mathbb{E}_{V_b}[f|_{V_b}].$$

□

Lemma 4.2.6. *Consider a cartesian diagram of admissible $\mathbb{Z}$-sets*

$$\begin{array}{ccc} V_1 & \xrightarrow{\varphi} & V_2 \\ \downarrow & & \downarrow \\ B_1 & \xrightarrow{\psi} & B_2 \end{array}$$

and suppose $V_2 \to B_2$ admits a section. Then, for $f \in C(V_2, W(\mathbb{C}))$,

$$\psi^* \mathbb{E}[f] = \mathbb{E}[\varphi^* f].$$

Proof. We have

$$\psi^* \mathbb{E}_{V_2/B_2}[f] = \frac{\psi^* \int_{V_2/B_2} f}{\psi^*[V_2/B_2]} = \frac{\int_{V_1/B_1} \varphi^* f}{[V_1/B_1]} = \mathbb{E}_{V_1/B_1}[\varphi^* f],$$

where the second equality is by Lemma 4.2.2 (note $[V/B] = \int_{V/B} 1$). □

Lemma 4.2.7. *For $f \in C(V, W(\mathbb{C}))$,*

$$\left(\mathbb{E}_{V/B}[f]\right)_{B_{\mathbf{k}}} = \mathbb{E}_{V_{\mathbf{k}}/B_{\mathbf{k}}}[f_{V_{\mathbf{k}}}].$$

Proof. Identifying $B_{\mathbf{k}} = (B \times \mathbf{k})_1$ as in 4.2.2.3 and recalling the definition in 4.2.3.2, we apply Lemma 4.2.6 to see

$$\left(\mathbb{E}_{V/B}[f]\right)_{B_{\mathbf{k}}} = \left(\pi_B^* \mathbb{E}_{V/B}[f]\right)\big|_{(B\times\mathbf{k})_1} = \left(\mathbb{E}_{V\times\mathbf{k}/B\times\mathbf{k}}[\pi_V^* F]\right)\big|_{(B\times\mathbf{k})_1}$$

where $\pi_B : B \times \mathbf{k} \to B$ and $\pi_V : V \times \mathbf{k} \to V$ are the projections. We then conclude by comparing the values at points with $\mathbb{E}_{V_{\mathbf{k}}/B_{\mathbf{k}}}[F_{V_{\mathbf{k}}}]$ using Lemma 4.2.5. □

4.3 Point counting motivic Euler products

In this section, we define motivic Euler products for morphisms of admissible $\mathbb{Z}$-sets and then explain how to compute them using classical Euler products.

4.3.1 Motivic Euler products

Recall from Section 4.2.2 that, for V an admissible $\mathbb{Z}$-set, we have the associated λ-ring $C(V, W(\mathbb{C}))$. Recall from Section 4.2.1 that we can then construct the filtered λ-ring of formal power series $C(V, W(\mathbb{C}))[[\underline{t}_{\mathbb{N}}]]$ with its associated σ-exponential Exp_σ and σ-logarithm Log_σ.

Definition 4.3.1 (Motivic Euler products)**.** For $V \to B$ a morphism of admissible $\mathbb{Z}$-sets and $H \in 1 + \mathrm{Fil}^1 C(V, W(\mathbb{C}))[[\underline{t}_{\mathbb{N}}]]$, we define

$$\prod_{V/B} H := \mathrm{Exp}_\sigma\left(\int_{V/B} \mathrm{Log}_\sigma(H)\right) \in C(B, W(\mathbb{C}))[[\underline{t}_{\mathbb{N}}]].$$

When $V \to \mathbf{1}$ is the final morphism we write

$$\prod_V H = \prod_{V/\mathbf{1}} H = \mathrm{Exp}_\sigma\left(\int_V \mathrm{Log}_\sigma(H)\right).$$

Remark 4.3.2. The relation between this formula and the motivic Euler products of [Bil23] will be detailed in [BDH25], explaining the nomenclature. In the present work, we do not need to make this connection explicit since it will suffice for our purposes to have the formula relating motivic Euler products to classical Euler products given in Proposition 4.3.7 below (which is reproved from a different perspective in [BDH25]).

Example 4.3.3. Let $H \in W(\mathbb{C})[[\underline{t}_{\mathbb{N}}]]$ with constant coefficient 1. Then we find

$$\begin{aligned} \prod_V H &= \mathrm{Exp}_\sigma\left(\int_V \mathrm{Log}_\sigma(H)\right) && \text{by definition} \\ &= \mathrm{Exp}_\sigma\left(\mathrm{Log}_\sigma(H)\left(\int_V 1\right)\right) && \text{by linearity of } \int_V \\ &= \mathrm{Exp}_\sigma\left(\mathrm{Log}_\sigma(H)[V]\right) && \text{by Example 4.2.1} \\ &= H^{[V]} && \text{by definition} \end{aligned}$$

as one hopes by the notation! Note that we have implicitly pulled back the coefficients of H from the final object $\mathbf{1}$ to get an element of $C(V, W(\mathbb{C}))[[\underline{t}_{\mathbb{N}}]]$ — in particular, when we are viewing H as an element of $C(V, W(\mathbb{C}))[[\underline{t}_{\mathbb{N}}]]$ here, it is *not* as the constant function on V with values H, but rather the function that takes value $p_i * H$ on each degree i point, where $*$ denotes the coefficient-wise pre-λ-ring structure as in 4.2.1.6 (see also [How24, Example 5.3.6 and subsequent warning]).

Motivic Euler products can be computed fiberwise (recall we defined fibers in 4.2.2.2):

Lemma 4.3.4. *Let $V \to B$ be a morphism of admissible $\mathbb{Z}$-sets. For any $b \in B$ and $H \in 1 + \mathrm{Fil}^1 C(V, W(\mathbb{C}))[[\underline{t}_{\mathbb{N}}]]$,*

$$\left(\prod_{V/B} H\right)(b) = \prod_{V_b}(H|_{V_b}),$$

where evaluation at b and restriction on power series are evaluated coefficient-wise.

Proof.

$$\begin{aligned} \left(\prod_{V/B} H\right)(b) &= \mathrm{Exp}_\sigma\left(\int_{V/B} \mathrm{Log}_\sigma(H)\right)(b) && \text{by definition} \\ &= \mathrm{Exp}_\sigma\left(\left(\int_{V/B} \mathrm{Log}_\sigma(H)\right)(b)\right) && \text{because evaluation is a map of pre-}\lambda\text{-rings} \\ &= \mathrm{Exp}_\sigma\left(\int_{V_b} \mathrm{Log}_\sigma(H)|_{V_b}\right) && \text{because integration is defined fiberwise} \\ &= \mathrm{Exp}_\sigma\left(\int_{V_b} \mathrm{Log}_\sigma(H|_{V_b})\right) && \text{because the restriction is a map of pre-}\lambda\text{-rings} \\ &= \prod_{V_b} H|_{V_b} && \text{by definition.} \end{aligned}$$

□

Motivic Euler products also behave well with respect to cartesian pullback.

Lemma 4.3.5. *Consider a cartesian diagram of admissible $\mathbb{Z}$-sets*

$$\begin{array}{ccc} V_1 & \xrightarrow{\varphi} & V_2 \\ \downarrow & & \downarrow \\ B_1 & \xrightarrow{\psi} & B_2 \end{array}$$

For $H \in C(V_2, W(\mathbb{C}))[[\underline{t}_{\mathbb{N}}]]$,

$$\psi^* \prod_{V_2/B_2} H = \prod_{V_1/B_1} \varphi^* H,$$

where the pullback functors are applied to power series coefficient-wise.

Proof. We have

$$\begin{aligned}
\psi^*\left(\prod_{V_2/B_2} H\right) &= \psi^*\left(\mathrm{Exp}_\sigma\left(\int_{V_2/B_2} \mathrm{Log}_\sigma(H)\right)\right) && \text{by definition}\\
&= \mathrm{Exp}_\sigma\left(\psi^* \int_{V_2/B_2} \mathrm{Log}_\sigma(H)\right) && \text{because } \psi^* \text{ is a map of pre-}\lambda\text{-rings}\\
&= \mathrm{Exp}_\sigma\left(\int_{V_1/B_1} \varphi^* \mathrm{Log}_\sigma(H)\right) && \text{by Lemma 4.2.2}\\
&= \mathrm{Exp}_\sigma\left(\int_{V_1/B_1} \mathrm{Log}_\sigma(\varphi^* H)\right) && \text{because } \varphi^* \text{ is a map of pre-}\lambda\text{-rings}\\
&= \prod_{V_1/B_1} \varphi^* H && \text{by definition.}
\end{aligned}$$

□

Combing these two lemmas, we deduce:

Lemma 4.3.6. *Let $V \to B$ be a map of admissible $\mathbb{Z}$-sets, and let $H \in C(V, W(\mathbb{C}))[[\underline{t}_{\mathbb{N}}]]$. For any $k \geq 1$,*

$$\left(\prod_{V/B} H\right)_{B_{\mathbf{k}}} = \prod_{V_{\mathbf{k}}/B_{\mathbf{k}}} H_{V_{\mathbf{k}}}$$

where the restriction $H_{V_{\mathbf{k}}}$ is formed coefficient-wise as in 4.2.3.2.

Proof. Identifying $B_{\mathbf{k}} = (B \times \mathbf{k})_1$ as in 4.2.2.3 and using the definition in 4.2.3.2, we apply Lemma 4.3.5 to see

$$\left(\prod_{V/B} H\right)_{B_{\mathbf{k}}} = \left(\pi_B^* \prod_{V/B} H\right)\Bigg|_{(B\times\mathbf{k})_1} = \left(\prod_{V\times\mathbf{k}/B\times\mathbf{k}} \pi_V^* H\right)\Bigg|_{(B\times\mathbf{k})_1}$$

where $\pi_B : B \times \mathbf{k} \to B$ and $\pi_V : V \times \mathbf{k} \to V$ are the projections. We then conclude by comparing the values at points with $\prod_{V_{\mathbf{k}}/B_{\mathbf{k}}} F_{V_{\mathbf{k}}}$ using Lemma 4.3.4. □

4.3.2 Evaluation of motivic Euler products using classical Euler products

The following proposition gives a description of motivic Euler products in terms of classical Euler products of series in $\mathbb{C}[[\underline{t}_{\mathbb{N}}]]$. In light of Example 4.3.3, it is a generalization of [How24, Proposition 5.2.7]. To state it, for H a power series with coefficients in $W(\mathbb{C})$ and $i \geq 1$, we write H_i for the power series $\mathbb{C}[[\underline{t}_{\mathbb{N}}]]$ obtained by taking the i^{th} ghost coordinate of all coefficients (note $H \mapsto H_i$ is a ring homomorphism).

In the following, we use the convention for restriction from V to $V_{\mathbf{k}}$ as in 4.2.3.2 (applied coefficient-wise to power series).

Proposition 4.3.7. *Let V be an admissible $\mathbb{Z}$-set and let $H \in C(V, W(\mathbb{C}))[[\underline{t}_{\mathbb{N}}]]$.*

$$\left(\prod_V H\right)_k = \prod_{|v|\in|V_{\mathbf{k}}|} H(|v|)_1(\underline{t}^{\deg(v)}) \tag{4.3.1}$$
$$= \prod_{|v|\in|V|} \left(H(|v|)_{k/\gcd(k,\deg(v))}(\underline{t}^{\deg(v)/\gcd(k,\deg(v))})\right)^{\gcd(k,\deg(v))}.$$

Remark 4.3.8. Comparing the expressions in Proposition 4.3.7 for $\left(\prod_V H\right)_k$ and $\left(\prod_{V_k} H\right)_1$, one finds that they agree. This is implied already by Lemma 4.3.6.

Example 4.3.9. In the case that the admissible $\mathbb{Z}$-set is $Y(\overline{\mathbb{F}}_q)$ for $Y/\mathbb{F}_q$ a variety and H is a power series whose coefficients are the classes associated to varieties $X_{\underline{j}}/Y$, $[X_{\underline{j}}(\overline{\mathbb{F}}_q)/Y(\overline{\mathbb{F}}_q)]$ as in [How24, Example 5.3.5], Proposition 4.3.7 says

$$\left(\prod_{Y(\overline{\mathbb{F}}_q)} \sum_{\underline{j}} [X_{\underline{j}}(\overline{\mathbb{F}}_q)/V(\overline{\mathbb{F}}_q)]\underline{t}^{\underline{j}}\right)_k = \prod_{y\in|Y_{\mathbb{F}_{q^k}}|} \sum_{\underline{j}} \#X_{\underline{j},\kappa(y)}(\kappa(y))\underline{t}^{\underline{j}\deg(y)}$$

where $\kappa(y)$ denotes the residue field at the closed point y of $Y_{\mathbb{F}_{q^k}}$.

Proof of Proposition 4.3.7. The equality of the two infinite products on the right of Equation (4.3.1) follows from the definition of the restriction of the coefficients of H in 4.2.3.2 and the fact that each orbit of degree d in V splits into $\gcd(d,k)$ orbits of degree $d/\gcd(d,k)$ in V_k.

We now show these infinite products agree with the k^{th} component of the motivic Euler product. We first note

$$\operatorname{Exp}_\sigma\left(\int_V \operatorname{Log}_\sigma(H)\right) = \operatorname{Exp}_\sigma\left(\sum_{|v|\in|V|} (\operatorname{Log}_\sigma(H(|v|))(t^{\deg(v)})\right)$$
$$= \prod_{|v|\in|V|} \operatorname{Exp}_\sigma\left(\operatorname{Log}_\sigma(H(|v|))(t^{\deg(v)})\right).$$
$$= \prod_{|v|\in|V|}\left(\prod_{|v|} H|_{|v|}\right)$$

where, in the final line, the first product is a usual infinite product over the countable set of orbits $|V|$, and the second product is a motivic Euler product over the orbit $|v|$, viewed itself as an admissible $\mathbb{Z}$-set.

Thus it suffices to assume $V = \mathbf{d}$ is a single orbit. In this case we are trying to compute the k^{th} component of

$$\prod_{\mathbf{d}} H = \mathrm{Exp}_\sigma\left(\int_{\mathbf{d}} \mathrm{Log}_\sigma(H)\right) = \mathrm{Exp}_\sigma\left(\left(\mathrm{Log}_\sigma(H(\mathbf{d}))\right)(t^d)\right)$$

where the substitution of t^d for an element of $W(\mathbb{C})$ is as in 4.2.1.3 and here it is performed coefficient-wise (note that we are writing t for the variable in $W(\mathbb{C}) = 1 + t\mathbb{C}[[t]]$ while we write t_i for the power series variables!), and the computation of $\int_{\mathbf{d}}$ follows from the definition ([How24, Definition 5.3.2]).

Thus, for $L := \mathrm{Log}_\sigma(H(\mathbf{d}))$ and $M := L(t^d)$, we are trying to compute $\mathrm{Exp}_\sigma(M)$. We note that, by Equation (4.2.2), $M_j = dL_{j/d}$ — in particular, this is zero for $d \nmid j$. Let $\mu = \frac{\mathrm{lcm}(k,d)}{k} = \frac{d}{\gcd(k,d)}$ and let $\nu = \frac{\mathrm{lcm}(k,d)}{d} = \frac{k}{\gcd(k,d)}$. Using the expansion of Exp_σ in [How24, Lemma 2.5.4], we find (below we note that $*$ is the coefficient wise pre-λ structure as in 4.2.1.6):

$$\begin{aligned}
\mathrm{Exp}_\sigma\left(L(t^d)\right)_k &= \prod_{j\geq 1} \exp\left(\frac{p_j \circ M}{j}\right)_k \\
&= \prod_{j\geq 1} \exp\left(\frac{(p_j \circ M)_k}{j}\right) \\
&= \prod_{j\geq 1} \exp\left(\frac{(p_j \circ p_k * M)_1}{j}\right) \\
&= \prod_{i\geq 1} \exp\left(\frac{(p_{i\mu} \circ p_k * M)_1}{i\mu}\right).
\end{aligned}$$

Here in the second equality we have used that passing to ghost components commutes with ring operations, and in the fourth equality we have used that $M_n = 0$ if $d \nmid n$. Continuing by factoring out a $p_i \circ$ in each term, we obtain

$$\begin{aligned}
&= \prod_{i\geq 1} \exp\left(\frac{(p_i \circ (p_\mu \circ p_k * M))_1}{i\mu}\right) \\
&= \prod_{i\geq 1} \exp\left(\frac{(p_i \circ (p_{k\mu} * M(\underline{t}^\mu)))_1}{i\mu}\right) \\
&= \prod_{i\geq 1} \exp\left(\frac{(p_i \circ (p_{k\mu/d} * L)(\underline{t}^\mu))}{i\mu/d}\right)_1 \\
&= \prod_{i\geq 1} \exp\left(\frac{d}{\mu}\frac{p_i \circ (p_\nu * L)(\underline{t}^\mu)}{i}\right)_1
\end{aligned}$$

where on the second line we have used that $p_\mu\circ$ acts as $p_\mu*$ followed by substitution of t_i^μ for t_i. Continuing by pulling out the integer multiple from the exponential as a power, we obtain

$$\begin{aligned}
&= \prod_{i\geq 1} \exp\left(\frac{p_i\circ(p_\nu * L)(\underline{t}^\mu)}{i}\right)_1^{\frac{d}{\mu}}\\
&= \left(\mathrm{Exp}_\sigma(p_\nu * L(\underline{t}^\mu))^{d/\mu}\right)_1\\
&= \left(p_\nu * \mathrm{Exp}_\sigma(L)(\underline{t}^\mu)^{d/\mu}\right)_1\\
&= \left(p_\nu * H(\mathbf{d})(\underline{t}^\mu)\right)_1^{d/\mu}\\
&= (H(\mathbf{d})_\nu(\underline{t}^\mu))^{d/\mu}.
\end{aligned}$$

Now, $\mathbf{d}_\mathbf{k}$ consists of $d/\mu = \gcd(k,d)$ orbits of degree μ, and $\nu = k/\gcd(k,d)$, so we conclude. □

4.4 Equidistribution and independence

In this section, we define our notion of equidistribution (Definition 4.4.1), then prove our main abstract result on the computation of asymptotic moment-generating functions in the presence of equidistribution, Theorem 4.4.3.

4.4.1 Equidistributing families

Definition 4.4.1. Let $A\to B$ be a morphism of admissible $\mathbb{Z}$-sets admitting a section $B\to A$. Let I be a directed set, and for each $d\in I$, suppose given an admissible $\mathbb{Z}$-set U_d and a morphism $\mathrm{ev}_d : U_d\times B\to A$ of admissible $\mathbb{Z}$-sets over B.

- For any positive integer k and admissible $\mathbb{Z}$-subset $B'\subseteq B_\mathbf{k}$, we define

$$\begin{aligned}
\mathrm{ev}_{d,B'} : U_{d,\mathbf{k}}(\mathbf{1}) &\longrightarrow \mathrm{Hom}_{B_\mathbf{k}}(B', A_\mathbf{k}) = \prod_{|b|\in|B'|}\mathrm{Hom}_{B_\mathbf{k}}(|b|, A_\mathbf{k})\\
u &\longmapsto \left(b\mapsto \mathrm{ev}_{d,\mathbf{k}}(u,b)\right).
\end{aligned}$$

  where, in the bottom formula, u is viewed as an element of $U_{d,\mathbf{k}}$.

- We say (U_d, ev_d) equidistributes on A/B if, for any $k\geq 1$ and any non-empty admissible $\mathbb{Z}$-subset $B'\subseteq B_\mathbf{k}$ of finite cardinality,

$$\lim_{d\in I}\left((\mathrm{ev}_{d,B'})_*\mu_{U_{d,\mathbf{k}}(\mathbf{1})}\right) = \mu_{\mathrm{Hom}_{B_\mathbf{k}}(B',A_\mathbf{k})} \tag{4.4.1}$$

  where, for any finite set Z, μ_Z is the uniform probability measure on Z.

Example 4.4.2 (Poonen's Bertini)**.** Let Y be a smooth, quasi-projective subscheme of $\mathbb{P}^n_{\mathbb{F}_q}$. Write S for $\mathbb{F}_q[x_0,\dots,x_n]$ and S_d for the set of degree d homogeneous polynomials in S. For any field extension $L/\mathbb{F}_q$, write $S(L)$ and $S_d(L)$ for $S\otimes_{\mathbb{F}_q} L$ and $S_d\otimes_{\mathbb{F}_q} L$, respectively. For each point $P\in\mathbb{P}^n(\overline{\mathbb{F}}_q)$, we fix a j_P such that x_{j_P} does not vanish at P; we make this choice so that j_P is constant on orbits. For $F\in S_d(\overline{\mathbb{F}}_q)$, we write F_P for the image of $F/x^d_{j_P}$ in $\mathcal{O}_{\mathbb{P}^n_{\overline{\mathbb{F}}_q},P}/\mathfrak{m}_P^2$.

Set $B=\mathbb{P}^n_{\mathbb{F}_q}(\overline{\mathbb{F}}_q)$ and for each $P\in B$, let A_P be the subset of $\mathcal{O}_{\mathbb{P}^n_{\overline{\mathbb{F}}_q},P}/\mathfrak{m}_P^2$ such that

$$A_P=\begin{cases}\{g_P\in\mathcal{O}_{\mathbb{P}^n_{\overline{\mathbb{F}}_q},P}/\mathfrak{m}_P^2 \mid \text{image of } g_P \text{ in } \mathcal{O}_{Y_{\overline{\mathbb{F}}_q},P}/\mathfrak{m}_P^2 \text{ is nonzero}\} & P\in Y\\ \mathcal{O}_{\mathbb{P}^n_{\overline{\mathbb{F}}_q},P}/\mathfrak{m}_P^2 & P\notin Y\end{cases}$$

Set $A=\bigsqcup_{P\in B}A_P$. Let U_d be the set of $F\in S_d(\overline{\mathbb{F}}_q)$ such that for all $P\in B$, the image of F in $\mathcal{O}_{\mathbb{P}^n_{\overline{\mathbb{F}}_q},P}/\mathfrak{m}_P^2$ lies in A_P; in other words, these are the polynomials such that their scheme-theoretic vanishing set $V(F)$ intersects Y transversely. Each of A, B, and U_d are admissible $\mathbb{Z}$-sets with the geometric Frobenius action.

Let $A\to B$ be the map $F_P\mapsto P$. For an admissible $\mathbb{Z}$-subset $B'\subseteq B_{\mathbf{k}}$ of finite cardinality, we have

$$\mathrm{Hom}_{B_{\mathbf{k}}}(B',A_{\mathbf{k}})=\prod_{|P|\in|B'|}A_{|P|}$$

where $|B'|$ is naturally viewed as a subset of $|\mathbb{P}^n_{\mathbb{F}_{q^k}}|$ and $A_{|P|}$ is canonically identified with a subset of $\mathcal{O}_{\mathbb{P}^n_{\mathbb{F}_{q^k}},|P|}/\mathfrak{m}_{|P|}^2$.

Viewing $U_{d,\mathbf{k}}(\mathbf{1})$ as the set of $F\in S_d(\mathbb{F}_{q^k})$ such that the image of F in $\mathcal{O}_{Y_{\mathbb{F}_{q^k}},|P|}/\mathfrak{m}_{|P|}^2$ is nonzero for all $|P|\in|Y_{\mathbb{F}_{q^k}}|$, the map $\mathrm{ev}_{d,B'}$ sends F to the tuple $(F_{|P|})_{|P|\in|B'|}$. It is a consequence of Poonen's sieve as in [Poo04] that (U_d,ev_d) equidistributes on A/B; this will be explained in greater generality in Section 4.5.1.

4.4.2 Asymptotic σ-moment-generating functions

Suppose (U_d,ev_d) equidistributes in A/B as in Definition 4.4.1. Given a function $\mathcal{X}\in C(A,W(\mathbb{C}))$, for any $b\in B$, we obtain a random variable on the fiber $U_d\times b$ (with $\mathbb{Z}$-action multiplied by $k=\deg(b)$, i.e. $U_{d,\mathbf{k}}$) by restricting $\mathcal{X}_d:=\mathrm{ev}_d^*\mathcal{X}$. Thus we may view $\mathcal{X}_d$ as a family of random variables on U_d parameterized by B, and then take their "sum" by integrating over B to obtain $X_d:=\int_{B\times U_d/U_d}\mathcal{X}_d$. The term equidistribution suggests that, as $d\to\infty$, the random variables in the family $\mathcal{X}_d$ will behave as if they are independent, so that one expects the moment-generating function of this "sum" X_d to approach the "product" of the moment-generating functions of the random variables in the family. Moreover, one expects to be able to compute the terms in this "product": the moment-generating functions of the random variable $\mathcal{X}_d|_{U_d\times b}$ should converge as $d\to\infty$ to the moment-generating function of $\mathcal{X}|_{A_b}$. The following result makes this intuition precise using the point-counting motivic Euler products of Section 4.3:

Theorem 4.4.3. *Let $A\to B$ be a morphism of admissible $\mathbb{Z}$-sets admitting a section. Suppose given a directed set I and for each $d\in I$, suppose given an admissible $\mathbb{Z}$-set U_d and a morphism $\mathrm{ev}_d : U_d\times B\to A$ of admissible $\mathbb{Z}$-sets over B. If (U_d,ev_d) equidistributes on A/B (Definition 4.4.1) then, for any $\mathcal{X}\in C(A,W(\mathbb{C}))$, letting*

$$\mathcal{X}_d := \mathrm{ev}_d^*\mathcal{X}\in C(U_d\times B, W(\mathbb{C})) \;\; \text{and} \;\; X_d := \int_{U_d\times B/U_d}\mathcal{X}_d\in C(U_d, W(\mathbb{C})),$$

we have

$$\lim_{d\in I}\mathbb{E}_{U_d}[\mathrm{Exp}_\sigma(X_d h_1)] = \lim_{d\in I}\mathbb{E}_{U_d}\left[\prod_{U_d\times B/U_d}\mathrm{Exp}_\sigma(\mathcal{X}_d h_1)\right] \tag{4.4.2}$$

$$= \prod_B \lim_{d\in I}\mathbb{E}_{U_d\times B/B}[\mathrm{Exp}_\sigma(\mathcal{X}_d h_1)] \tag{4.4.3}$$

$$= \prod_B \mathbb{E}_{A/B}\left[\mathrm{Exp}_\sigma(\mathcal{X} h_1)\right]. \tag{4.4.4}$$

Proof. We first note that

$$\begin{aligned}\mathrm{Exp}_\sigma(X_d h_1) &= \mathrm{Exp}_\sigma\left(\int_{U_d\times B/U_d}\mathcal{X}_d h_1\right)\\ &= \mathrm{Exp}_\sigma\left(\int_{U_d\times B/U_d}\mathrm{Log}_\sigma(\mathrm{Exp}_\sigma(\mathcal{X}_d h_1))\right)\\ &= \prod_{U_d\times B/U_d}\mathrm{Exp}_\sigma(\mathcal{X}_d h_1).\end{aligned}$$

In particular, from this identity we obtain the first equality Equation (4.4.2).

To obtain the next two equalities, Equation (4.4.3) and Equation (4.4.4), we argue on the k^{th} component for each k. It follows from Lemma 4.2.3 that we can compute the k^{th} ghost component as

$$\left(\mathbb{E}_{U_d}\left[\prod_{U_d\times B/U_d}\mathrm{Exp}_\sigma(\mathcal{X}_d h_1)\right]\right)_k = \mathbb{E}_{U_{d,\mathbf{k}}(\mathbf{1})}\left[\left(\mathrm{res}_k\prod_{U_d\times B/U_d}\mathrm{Exp}_\sigma(\mathcal{X}_d h_1)\right)_1\right].$$

For $u\in U_{d,\mathbf{k}}(\mathbf{1})$, we have

$$\begin{aligned}\left(\mathrm{res}_k\prod_{U_d\times B/U_d}\mathrm{Exp}_\sigma(\mathcal{X}_d h_1)\right)(u) &= \left(\prod_{U_d\times B/U_d}\mathrm{Exp}_\sigma(\mathcal{X}_d h_1)\right)_{U_{d,\mathbf{k}}}(u)\\ &= \left(\prod_{U_{d,\mathbf{k}}\times B_{\mathbf{k}}/U_{d,\mathbf{k}}}\mathrm{Exp}_\sigma(\mathcal{X}_{d,U_{d,\mathbf{k}}\times B_{\mathbf{k}}} h_1)\right)(u)\\ &= \prod_{B_{\mathbf{k}}}\mathrm{Exp}_\sigma(\mathrm{ev}_{d,\mathbf{k}}(u,-)^*\mathcal{X}_{A_{\mathbf{k}}} h_1).\end{aligned}$$

where the first equality is by definition, the second equality is by Lemma 4.3.6, and the third equality follows from writing $\mathcal{X}_{d,U_{d,\mathbf{k}}\times B_{\mathbf{k}}} = \mathrm{ev}^*_{d,\mathbf{k}}\mathcal{X}_{A_{\mathbf{k}}}$ and Lemma 4.3.4.

Passing to the first ghost component, we obtain, by Proposition 4.3.7,

$$\left(\mathrm{res}_k \prod_{U_d\times B/U_d} \mathrm{Exp}_\sigma(\mathcal{X}_d h_1)\right)_1 (u) = \prod_{|b|\in|B_{\mathbf{k}}|} \mathrm{Exp}_\sigma(\mathcal{X}_{A_{\mathbf{k}}}(\mathrm{ev}_{d,\mathbf{k}}(u,b))p_{\deg(b)})_1,$$

where to obtain the power sum monomial $p_{\deg(b)}$ we have used that $h_1(\underline{t}^n) = p_n$.

Now, note that for each monomial symmetric function $m_\tau \in \Lambda$, only the finitely many $|b| \in |B_{\mathbf{k}}|$ of degree less than $|\tau|$ can contribute to the coefficient of m_τ in the product on the right. Let B' be the set of these, so that the coefficient of m_τ in this product is the same as its coefficient in

$$\prod_{|b|\in|B'|} \mathrm{Exp}_\sigma(\mathcal{X}_{A_{\mathbf{k}}}(\mathrm{ev}_{d,\mathbf{k}}(u,b))p_{\deg(b)})_1.$$

Now, since (U_d, ev_d) equidistributes on A/B,

$$\begin{aligned}\lim_{d\in I} \mathbb{E}_{U_{d,\mathbf{k}}(\mathbf{1})}\left[\prod_{|b|\in|B'|} \mathrm{Exp}_\sigma(\mathcal{X}_{A_{\mathbf{k}}}(\mathrm{ev}_{d,\mathbf{k}}(u,b))p_{\deg(b)})_1\right]& \\ = \mathbb{E}_{\mathrm{Hom}_{B_{\mathbf{k}}}(B',A_{\mathbf{k}})}\left[\prod_{|b|\in|B'|} \mathrm{Exp}_\sigma\left(\mathcal{X}_{A_{\mathbf{k}}}(\varphi(|b|))p_{\deg(b)}\right)_1\right]&, \end{aligned} \quad (4.4.5)$$

where φ on the right denotes the varying element of $\mathrm{Hom}_{B_{\mathbf{k}}}(B', A_{\mathbf{k}})$ with respect to which we are taking expectation. Since

$$\mathrm{Hom}_{B_{\mathbf{k}}}(B', A_{\mathbf{k}}) = \prod_{|b|\in|B'|} \mathrm{Hom}_{B_{\mathbf{k}}}(|b|, A_{\mathbf{k}}),$$

the right-hand side of Equation (4.4.5) is equal to

$$\prod_{|b|\in|B'|} \mathbb{E}_{\mathrm{Hom}_{B_{\mathbf{k}}}(|b|,A_{\mathbf{k}})}\left[\mathrm{Exp}_\sigma\left(\mathcal{X}_{A_{\mathbf{k}}}(\varphi(|b|))p_{\deg(b)}\right)_1\right]. \quad (4.4.6)$$

Note that we have a natural identification

$$\mathrm{Hom}_{B_{\mathbf{k}}}(|b|, A_{\mathbf{k}}) = \mathrm{Hom}(\mathbf{1}, (A_{\mathbf{k}})_b).$$

Applying the $k = 1$ case of Lemma 4.2.3 to each term, we thus find that Equation (4.4.6) is equal to

$$\prod_{|b|\in|B'|} \left(\mathbb{E}_{(A_{\mathbf{k}})_b}\left[\mathrm{Exp}_\sigma(\mathcal{X}_{A_{\mathbf{k}}}|_{(A_{\mathbf{k}})_b} p_{\deg(b)})\right]\right)_1. \quad (4.4.7)$$

By Lemma 4.2.5, this is equal to

$$\prod_{|b|\in|B'|} \left(\mathbb{E}_{A_{\mathbf{k}}/B_{\mathbf{k}}}\left[\mathrm{Exp}_\sigma(\mathcal{X}_{A_{\mathbf{k}}} p_{\deg(b)})\right](|b|)\right)_1. \quad (4.4.8)$$

Note that, again for the coefficient of any fixed m_τ, Equation (4.4.8) agrees with the corresponding product over $|B_{\mathbf{k}}|$ whenever $|B'|$ is a sufficiently large subset of finite cardinality. Putting this all together, we obtain the desired equality:

$$\begin{aligned}\lim_{d\in I}\left(\mathbb{E}_{U_d}\left[\prod_{U_d\times B/U_d}\mathrm{Exp}_\sigma(\mathcal{X}_d h_1)\right]\right)_k &= \prod_{|b|\in|B_{\mathbf{k}}|}\left(\mathbb{E}_{A_{\mathbf{k}}/B_{\mathbf{k}}}\left[\mathrm{Exp}_\sigma(\mathcal{X}_{A_{\mathbf{k}}}p_{\deg(b)})\right](|b|)\right)_1\\ &= \prod_{|b|\in|B_{\mathbf{k}}|}\left(\left(\mathbb{E}_{A/B}\left[\mathrm{Exp}_\sigma(\mathcal{X}p_{\deg(b)})\right]\right)_{B_{\mathbf{k}}}(|b|)\right)_1\\ &= \left(\prod_B \mathbb{E}_{A/B}\left[\mathrm{Exp}_\sigma(\mathcal{X}h_1)\right]\right)_k\end{aligned}$$

where the second equality is by Lemma 4.2.7 and the third is by Proposition 4.3.7. □

Remark 4.4.4. Let $H\in 1+\mathrm{Fil}^1\mathbb{Z}[[\underline{t}_{\mathbb{N}}]]$. Then, either by the same argument as in the proof, or as a formal consequence of the equivalence of Λ-distributions deduced from the equality of σ-moment-generating functions (cf. [How24, §3.2]), in the setting of Theorem 4.4.3 one also obtains:

$$\begin{aligned}\lim_{d\in I}\mathbb{E}_{U_d}[H^{X_d}] &= \lim_{d\in I}\mathbb{E}_{U_d}\left[\prod_{U_d\times B/U_d}H^{X_d}\right]\\ &= \prod_{U_d\times B/U_d}\lim_{d\in I}\mathbb{E}_{U_d\times B/B}[H^{X_d}]\\ &= \prod_B \mathbb{E}_{A/B}\left[H^{\mathcal{X}}\right].\end{aligned}$$

The σ-moment-generating function corresponds to $H=\mathrm{Exp}_\sigma(h_1)=1+h_1+h_2+\ldots$, while the falling moment-generating function of [How24, Example 3.2.3-(2)] corresponds to $H=1+h_1$. Similarly, the proof can be extended to show one can compute joint moment-generating functions: e.g., for $\mathcal{X}$ and $\mathcal{Y}$ two families,

$$\lim_{d\in I}\mathbb{E}_{U_d}\left[H(\underline{t})^{X_d}H(\underline{s})^{Y_d}\right]=\prod_B\mathbb{E}_{A/B}\left[H(\underline{t})^{\mathcal{X}}H(\underline{s})^{\mathcal{Y}}\right].$$

4.5 Equidistribution for homogeneous polynomials

In this section we first use Theorem C of Chapter 2 to show equidistribution holds for homogeneous polynomials with certain natural conditions on their Taylor expansions (generalizing those accessible using [Poo04, Theorem 1.3]) — see Proposition 4.5.1. In Section 4.5.2 we explain how the computation of the σ-moment-generating functions for L-functions of Dirichlet characters made in [How24, Theorem B] can be handled by combining Proposition 4.5.1 and Theorem 4.4.3. In Section 4.5.3 we apply Proposition 4.5.1 and Theorem 4.4.3 to compute the asymptotic σ-moment-generating functions for zeta functions of hypersurfaces in a quasi-projective variety satisfying some exotic transversality conditions as in Example 2.5.2 of Chapter 2 — see Theorem 4.5.4.

4.5.1 Establishing equidistribution

Let $Y_1, \ldots, Y_u$ be quasi-projective subschemes of $\mathbb{P}^n_{\mathbb{F}_q}$ of dimensions $m_i = \dim Y_i$ with locally closed embeddings $\iota_1, \ldots, \iota_u$, respectively. For each i, let $\mathcal{Q}_i$ be a locally free quotient of $\iota_i^* \Omega^1_{\mathbb{P}^n}$ of rank $\ell_i \geq m_i$, and let $\mathcal{K}_i = \ker(\iota_i^* \Omega^1_{\mathbb{P}^n} \to \mathcal{Q}_i)$. Define

$$\mathcal{E}_i = \left(\iota_i^* \mathcal{P}^1(\mathcal{O}_{\mathbb{P}^n_{\mathbb{F}_q}})\right)/\mathcal{K}_i \quad \text{and} \quad \mathcal{E}_{i,d} = \left(\iota_i^* \mathcal{P}^1(\mathcal{O}_{\mathbb{P}^n_{\mathbb{F}_q}}(d))\right)/\mathcal{K}_i(d)$$

where $\mathcal{P}^1(\mathcal{F})$ is the sheaf of 1-principal parts of an $\mathcal{O}_{\mathbb{P}^n}$-module $\mathcal{F}$.

We write $x_0, \ldots, x_n$ for the homogeneous coordinates on $\mathbb{P}^n$. For every point $P \in \mathbb{P}^n(\overline{\mathbb{F}}_q)$, fix a nonnegative integer M_P and a j_P such that x_{j_P} is non-vanishing at P; we make these choices so that M_P and j_P are constant on orbits in $\mathbb{P}^n(\overline{\mathbb{F}}_q)$. Given $F \in S_d(\overline{\mathbb{F}}_q)$, write F_P for the image of $F/x_{j_P}^d$ in $\mathcal{O}_{\mathbb{P}^n_{\overline{\mathbb{F}}_q},P}/\mathfrak{m}_P^{M_P+1}$.

Let $\phi : \mathbb{P}^n_{\overline{\mathbb{F}}_q} \to \mathbb{P}^n_{\mathbb{F}_q}$ be the natural map.

Proposition 4.5.1. *With notation as above, set $B = \mathbb{P}^n_{\mathbb{F}_q}(\overline{\mathbb{F}}_q)$. For each $P \in B$, let A_P be a subset of $\mathcal{O}_{\mathbb{P}^n_{\overline{\mathbb{F}}_q},P}/\mathfrak{m}_P^{M_P+1}$ such that for all but finitely many P, A_P contains F_P for all homogeneous $F \in S(\overline{\mathbb{F}}_q)$ such that the image of F_P in $\phi^*\mathcal{E}_i|_P$ is nonzero for all i.*

Set $A = \bigsqcup_{P \in B} A_P$. Let U_d be the set of $F \in S_d(\overline{\mathbb{F}}_q)$ such that for all $P \in B$, the image F_P of F in $\mathcal{O}_{\mathbb{P}^n_{\overline{\mathbb{F}}_q},P}/\mathfrak{m}_P^{M_P+1}$ lies in A_P. Each of A, B, and U_d are admissible $\mathbb{Z}$-sets with the geometric Frobenius action.

Let $A \to B$ be the map $F_P \mapsto P$ and $\mathrm{ev}_d : U_d \times B \to A$ the map $(F, P) \mapsto F_P$. Then (U_d, ev_d) equidistributes on A/B in the sense of Definition 4.4.1.

Proof. For an admissible $\mathbb{Z}$-subset $B' \subseteq B_{\mathbf{k}}$ of finite cardinality, we have

$$\mathrm{Hom}_{B_{\mathbf{k}}}(B', A_{\mathbf{k}}) = \prod_{|P| \in |B'|} \mathrm{Hom}_{B_{\mathbf{k}}}(|P|, A_{\mathbf{k}}).$$

If we identify each orbit $|P| \subseteq B' \subseteq \mathbb{P}^n(\overline{\mathbb{F}}_q)$ with a closed point of $\mathbb{P}^n_{\mathbb{F}_{q^k}}$, then we obtain a canonical identification of $\mathrm{Hom}_{B_{\mathbf{k}}}(|P|, A_{\mathbf{k}})$ with a subset $A_{|P|}$ of $\mathcal{O}_{\mathbb{P}^n_{\mathbb{F}_{q^k}},|P|}/\mathfrak{m}_{|P|}^{M_P+1}$. Viewing $U_{d,\mathbf{k}}(\mathbf{1})$ as the set of $F \in S_d(\mathbb{F}_{q^k})$ such that the image $F_{|P|}$ of $F/x_{j_{|P|}}^d$ in $\mathcal{O}_{\mathbb{P}^n_{\mathbb{F}_{q^k}},|P|}/\mathfrak{m}_{|P|}^{M_{|P|}+1}$ lies in $A_{|P|}$ for all $|P| \in |\mathbb{P}^n_{\mathbb{F}_{q^k}}|$, the map $\mathrm{ev}_{d,B'}$ sends F to the tuple $(F_{|P|})_{|P| \in |B'|}$.

To establish Equation (4.4.1) it suffices to show equality for each singleton $\{(F_{|P|})_{|P|}\}$. On the right side we have

$$\mu_{\mathrm{Hom}_{B_{\mathbf{k}}}(B', A_{\mathbf{k}})}(\{(F_{|P|})_{|P|}\}) = \prod_{|P| \in |B'|} \frac{1}{\# A_{|P|}}.$$

The left side can be viewed as a conditional probability:

$$\begin{aligned}
&\big((\mathrm{ev}_{d,B'})_*\mu_{U_{d,\mathbf{k}}(\mathbf{1})}\big)(\{(F_{|P|})_{|P|}\}) \\
&\quad= \mu_{U_{d,\mathbf{k}}(\mathbf{1})}(\mathrm{ev}_{d,B'}^{-1}(\{(F_{|P|})_{|P|}\}) \\
&\quad= \frac{\#\{G \in S_d(\mathbb{F}_{q^k}) \mid G_{|P|} = F_{|P|} \text{ for all } |P| \in |B'| \text{ and } G \in U_{d,\mathbf{k}}(\mathbf{1})\}/\#S_d(\mathbb{F}_{q^k})}{\#U_{d,\mathbf{k}}(\mathbf{1})/\#S_d(\mathbb{F}_{q^k})}.
\end{aligned}$$

By Theorem C of Chapter 2, as $d \to \infty$, this converges to

$$\frac{\left(\prod_{|P|\in|B'|} \frac{1}{\#\left(\mathscr{O}_{\mathbb{P}^n_{\mathbb{F}_{q^k}}}/\mathfrak{m}_{|P|}^{M_{|P|}+1}\right)}\right)\left(\prod_{|P|\notin|B'|} \frac{\#A_{|P|}}{\#\left(\mathscr{O}_{\mathbb{P}^n_{\mathbb{F}_{q^k}}}/\mathfrak{m}_{|P|}^{M_{|P|}+1}\right)}\right)}{\prod_{|P|\in|\mathbb{P}^n_{\mathbb{F}_{q^k}}|} \frac{\#A_{|P|}}{\#\left(\mathscr{O}_{\mathbb{P}^n_{\mathbb{F}_{q^k}}}/\mathfrak{m}_{|P|}^{M_{|P|}+1}\right)}} = \prod_{|P|\in|B'|} \frac{1}{\#A_{|P|}}.$$

So Equation (4.4.1) is satisfied, and thus (U_d, ev_d) equidistributes on A/B. □

Remark 4.5.2. When each of the Y_i is smooth and $\mathscr{Q}_i = \Omega_{Y_i}$, one can invoke [Poo04, Theorem 1.3] instead of its generalization Theorem C of Chapter 2.

4.5.2 Example: *L*-functions of characters

We now explain how this framework leads to a more transparent computation of the asymptotic σ-moment-generating functions for L-functions of Dirichlet characters as in [How24, Theorem B].

Let κ be a finite field of cardinality q and fix a prime ℓ coprime to q, and a non-trivial order ℓ character χ of $\mu_\ell(\kappa)$. Let U_d be the space of ℓ-power free degree d polynomials in the variable x. As in [How24, §7.1], for $f \in U_d$, we obtain a Kummer character χ_f of $\mathrm{Gal}(\overline{\kappa(f)(x)}/\kappa(f)(x))$ sending σ to $\chi(\sigma(f^{1/\ell})/f^{1/\ell})$. One can compute the L-function (with no factor at ∞) as an Euler product

$$\mathcal{L}(\chi_f, t) = \prod_{|z|\in|\mathbb{A}^1_{\kappa(f)}|} \frac{1}{1-\chi(f(z)^{(\#\kappa(f,z)-1)/\ell})\, t^{\deg(z)}}, \tag{4.5.1}$$

where $\kappa(f,z)$ is the extension generated by z and the coefficients of κ and where we set $\chi(0) = 0$. We define a random variable X_d on U_d sending f to $\mathcal{L}(\chi_f, t)$.

We explain how this fits into the context of Proposition 4.5.1: for each $P \in \mathbb{A}^1(\overline{\kappa}) \subseteq \mathbb{P}^1(\overline{\kappa})$, we set $M_P = \ell - 1$, $j_P = 1$, and $A_P = \mathscr{O}_{\mathbb{P}^n_{\overline{\kappa}},P}/\mathfrak{m}_P^\ell - \{0\}$. For $\infty = [1:0]$, we set $M_\infty = 0$, $j_\infty = 0$, and $A_\infty = \{1\} \subseteq \mathscr{O}_{\mathbb{P}^n_{\overline{\kappa}},\infty}/\mathfrak{m}_\infty = \overline{\kappa}$. Then dividing by x_1^d identifies the U_d appearing in Proposition 4.5.1 with the set of ℓ-power free monic polynomials in $x = x_0/x_1$ that we have called U_d here, and under this identification the map ev_d is the natural map.

Now, for $P \in \mathbb{A}^1(\overline{\kappa})$, we define $\mathcal{X}_P$ to send a germ $g \in A_P$ to

$$\frac{1}{1-\chi(g(P)^{(\#\kappa(g)-1)/\ell})t} = [\chi(g(P)^{(\#\kappa(g)-1)/\ell})]$$

where $\kappa(g)$ is the extension of $\kappa(P)$ in $\overline{\kappa}$ generated by the coefficients of g. We define $\mathcal{X}_\infty$ to be the trivial random variable. It is a straightforward computation from Equation (4.5.1) that

$$X_d = \int_{U_d \times \mathbb{A}^1(\overline{\kappa})/U_d} \mathrm{ev}_d^* \mathcal{X},$$

thus Proposition 4.5.1 and Theorem 4.4.3 imply

$$\lim_{d\to\infty} \mathbb{E}[\mathrm{Exp}_\sigma(X_d h_1)] = \prod_{\mathbb{A}^1(\overline{\kappa})} \mathbb{E}_{A/\mathbb{A}^1(\overline{\kappa})}[\mathrm{Exp}_\sigma(\mathcal{X}_P h_1)],$$

where the term at ∞ has gone away because it is identically 1.

Now, we compute $\mathbb{E}[\mathrm{Exp}_\sigma(\mathcal{X}_P h_1)]$ for $P \in \mathbb{A}^1(\overline{\kappa})$: let χ_P denote the $\mathbb{C}$-valued function on A_P sending a germ g to $\chi(g(P)^{(\#\kappa(g)-1)/\ell})$ so that $\mathcal{X}_P = [\chi_P]$. Now, since $h_j \circ ([a]h_1) = [a^j]h_j$ for any $a \in \mathbb{C}$ (see [How24, Lemma 2.2.4]),

$$\mathrm{Exp}_\sigma(\mathcal{X}_P h_1) = \sum_{j\geq 0} [\chi_P^j] h_j.$$

We can compute the k^{th} component of $\mathbb{E}[[\chi_P^n]]$ using Lemma 4.2.3. The restriction of $[\chi_P]$ to $(A_P)_{\mathbf{k}}(\mathbf{1})$, i.e. to the set of germs with coefficients in the degree k extension of $\kappa(P)$ in $\overline{\kappa}$, sends g to $[\chi(g(P)^{(q_P^k-1)/\ell})]$, where $q_P = \#\kappa(P)$. Taking the first component, we find, $[\chi_P^n]_1 = [\chi_P]_1^n$ sends g to

$$\chi(g(P)^{n(q_P^k-1)/\ell}).$$

Thus the expectation is zero unless $\ell | n$ (since if $\ell \nmid n$ then every ℓ^{th} root of unity value is equally likely and the only other value it takes is zero), and when $\ell | n$ it is

$$\frac{(q_P^k)^{\ell-1}(q_P^k-1)-1}{(q_P^k)^\ell - 1} = \frac{1}{1+q_P^{-k}+\ldots+q_P^{(1-\ell)k}}$$

since the function is identically 1 when $g(P) \neq 0$ and 0 when $g(P) = 0$. Thus,

$$\mathbb{E}[\mathrm{Exp}_\sigma(\mathcal{X}_P h_1)] = 1 + \frac{1}{1+[q_P^{-1}]+\ldots+[q_P^{(1-\ell)}]} \sum_{j\geq 1} h_{\ell j}.$$

It follows that

$$\mathbb{E}_{A/\mathbb{A}^1(\overline{\kappa})}[\mathrm{Exp}_\sigma(\mathcal{X} h_1)]$$

is the pullback from a point of

$$1 + \frac{1}{1+[q^{-1}]+\ldots+[q^{(1-\ell)}]} \sum_{j\geq 1} h_{\ell j}.$$

Thus, applying Example 4.3.3,

$$\prod_{\mathbb{A}^1(\overline{\kappa})} \mathbb{E}_{A/\mathbb{A}^1(\overline{\kappa})}[\mathrm{Exp}_\sigma(\mathcal{X} h_1)] = \left(1 + \frac{1}{1 + [q^{-1}] + \ldots + [q^{(1-\ell)}]} \sum_{j \geq 1} h_{\ell j}\right)^{\mathbb{A}^1(\overline{\kappa})}$$
$$= \left(1 + \frac{1}{1 + [q^{-1}] + \ldots + [q^{(1-\ell)}]} \sum_{j \geq 1} h_{\ell j}\right)^{[q]}.$$

In particular, if we

(a) replace $\mathcal{L}$ with its reciprocal (as a power series in $1 + t\mathbb{C}[[t]]$, which is the negative for the additive structure of the Witt vectors), and

(b) scale the variables by $[q^{-1/2}]$,

then we recover the σ-moment-generating function described in [How24, Theorem B]. Indeed, by [How25, Theorem 2.2.1], replacing $\mathcal{L}$ with its reciprocal will replace the $h_{\ell j}$'s with $(-1)^{\ell j} e_{\ell j}$'s in the σ-moment-generating function, and scaling the random variable by $[z]$ for $z \in \mathbb{C}$ changes the σ-moment-generating function by scaling each of the variables t_i by $[z]$.

When $\ell \geq 3$, the more refined joint σ-moment-generating function between X_d and the random variable $\overline{X}_d$ obtained from the complex conjugate character to χ is of the most interest; this can be recovered similarly by using the natural extension of Theorem 4.4.3 to joint σ-moment-generating functions described in Remark 4.4.4.

4.5.3 Application: Zeta functions of hypersurface sections with exotic transversality conditions

We consider the setup of Example 2.5.2 of Chapter 2. Let κ be a finite field of cardinality q, let $Y \hookrightarrow \mathbb{P}^n_\kappa$ be a quasi-projective subscheme, let $W \hookrightarrow Y \times_\kappa \mathbb{P}^n_\kappa$ be a closed subscheme such that the projection from W to Y is smooth of relative dimension $\ell \geq \dim Y$ and such that the graph of $Y \hookrightarrow \mathbb{P}^n_\kappa$, $Y \to Y \times_\kappa \mathbb{P}^n_\kappa$, factors through W. In other words, $W \to Y$ is a smooth family of subvarieties of $\mathbb{P}^n_\kappa$ of at least the same dimension as Y such that, at each point $P \in Y(\overline{\kappa})$, the fiber W_P contains P.

We let U_d be the set of degree d homogeneous polynomials F in $n+1$ variables such that $V(F)$ is transverse to W_P at all points $P \in Y(\overline{\kappa}) \cap V(F)(\overline{\kappa})$. We write X_d for the random variable on U_d sending F to the zeta function

$$Z_{V(F)\cap Y_{\kappa(F)}/\kappa(F)}(t).$$

Example 4.5.3. If Y is smooth and $W = Y \times \mathbb{P}^n_\kappa$, then X_d is the random variable sending a smooth hypersurface section to its zeta function. On the other hand, Example 2.5.2 of Chapter 2 shows there are other geometrically interesting examples.

In light of Example 4.5.3 and [How25, Theorem 2.2.1], the following is a generalization of [How24, Theorem 8.3.1].

Theorem 4.5.4. *With notation as above, as $d \to \infty$, the Λ-distribution of X_d converges to a binomial Λ-distribution as in [How24, Definition 3.3.2] with parameters*

$$p = \frac{[q^{-1}] - [q^{-(\ell+1)}]}{1 - [q^{-(\ell+1)}]} = \frac{[q^{\ell}] - 1}{[q^{\ell+1}] - 1} \quad \text{and} \quad N = [Y(\overline{\kappa})].$$

Equivalently,

$$\lim_{d\to\infty} \mathbb{E}[\mathrm{Exp}_\sigma(X_d h_1)] = (1 + p(h_1 + h_2 + \ldots))^N.$$

Proof. We are in the setup of Proposition 4.5.1 with $M_P = 1$ for all P, with $u = 1$ and $\mathcal{Q} = \mathcal{Q}_1 = \gamma^*\Omega_{W/Y}$, where γ is the map $Y \to W$ induced by the graph of the immersion $Y \to \mathbb{P}^n_\kappa$. The j_P can be chosen arbitrarily; it will not affect the conditions below.

For $P \in Y(\overline{\kappa})$, we consider the random variable $\mathcal{X}_P$ on A_P sending a germ g to $\frac{1}{1-t}$ if $g(P) = 0$ and 0 otherwise, and for $P \in \mathbb{P}^n(\overline{\kappa}) - Y(\overline{\kappa})$, we set $\mathcal{X}_P$ to be the trivial random variable.

Then

$$X_d = \int_{U_d \times \mathbb{P}^n(\overline{\kappa})/U_d} \mathrm{ev}_d^* \mathcal{X}.$$

By Proposition 4.5.1, (U_d, ev_d) equidistributes on $A/\mathbb{P}^n(\overline{\kappa})$, thus, by Theorem 4.4.3, the asymptotic σ-moment-generating function is

$$\prod_{\mathbb{P}^n(\overline{\kappa})} \mathbb{E}_{A/\mathbb{P}^m(\overline{\kappa})}[\mathrm{Exp}_\sigma(\mathcal{X}h_1)] = \prod_{Y(\overline{\kappa})} \mathbb{E}_{A_{Y(\overline{\kappa})}/Y(\overline{\kappa})}[\mathrm{Exp}_\sigma(\mathcal{X}h_1)]$$

where the factor corresponding to $\mathbb{P}^n(\overline{\kappa}) - Y(\overline{\kappa})$ disappeared because the moment-generating function of the trivial random variable is identically 1.

For $P \in Y(\overline{\kappa})$ of degree k, $\mathcal{X}_P$ is a Bernoulli random variable equal to $\frac{1}{1-t}$ (the unit in $W(\mathbb{C})$) with probability

$$\frac{[q^{-k}] - [q^{-(\ell+1)k}]}{1 - [q^{-(\ell+1)k}]}.$$

This can be checked on each ghost component using Lemma 4.2.3; the numerator is the probability of a section vanishing at P and being transverse to W_P at P, whereas the denominator is the probability of a section either not vanishing or vanishing and being transverse.

It follows that $\mathrm{Exp}_\sigma(\mathcal{X}h_1)|_{Y(\overline{\kappa})}$ is the pullback of $(1 + p(h_1 + h_2 + \ldots))$ from $\mathbf{1}$ to $Y(\overline{\kappa})$. Thus, by Example 4.3.3,

$$\prod_{Y(\overline{\kappa})} \mathbb{E}_{A_{Y(\overline{\kappa})}/Y(\overline{\kappa})}[\mathrm{Exp}_\sigma(\mathcal{X}h_1)] = \prod_{Y(\overline{\kappa})} (1 + p(h_1 + h_2 + \ldots)) = (1 + p(h_1 + h_2 + \ldots))^{[Y(\overline{\kappa})]}.$$

□

Remark 4.5.5. For $F \in U_d$ as above, $V(F)$ may not intersect Y transversely, so there is not an obvious L-function to extract in order to give a result analogous to Theorem D. However, Proposition 4.5.1 is robust enough to allow one to additionally impose the condition that $V(F)$ intersect Y transversely, giving an interesting L-function. We leave the computation of the L-function Λ-distribution in this case to the interested reader, but note that the motivic Euler product for the zeta function random variable in this setting will not typically be expressible as a single pre-λ power: the pointwise moment-generating function at P will depend on the dimension of the intersection of the tangent space of Y at P and the tangent space of W_P at P.

4.6 Equidistribution for tuples of homogeneous polynomials

In this section we use the results of [BK12] to show equidistribution holds for tuples of homogeneous polynomials intersecting a fixed smooth quasi-projective variety transversely (Proposition 4.6.1), then combine this with Theorem 4.4.3 to prove Theorem D. As in the proof of [How24, Theorem C], we first study the geometric random variable that sends a complete intersection to its zeta function (Theorem 4.6.2; cf. [How24, Theorem 8.3.1]), then argue with basic properties of independence to obtain the computation of the asymptotic moment-generating function in Theorem D.

4.6.1 Establishing equidistribution

4.6.1.1 For $\underline{d} = (d_1, \dots, d_r)$ a tuple of positive integers, write $S_{\underline{d}}$ for the product $S_{d_1} \times \dots \times S_{d_r}$ and identify it with the global sections of $\mathscr{O}_{\mathbb{P}^n_{\mathbb{F}_q}}(\underline{d}) = \bigoplus_{i=1}^r \mathscr{O}_{\mathbb{P}^n_{\mathbb{F}_q}}(d_i)$.

4.6.1.2 Fix non-negative integers m and r. Set $I_{m,r} = \mathbb{N}^r$ with an ordering described as follows: for tuples $\underline{a} = (a_1, \dots, a_r)$ and $\underline{b} = (b_1, \dots, b_r)$, $\underline{a} \leq \underline{b}$ if and only if $a_i \leq b_i$ for all i and $\max(b_i)^{-m} q^{\min(b_i)/(m+1)} \leq \max(a_i)^{-m} q^{\min(a_i)/(m+1)}$.

4.6.1.3 For every point $P \in \mathbb{P}^n(\overline{\mathbb{F}}_q)$, fix a non-vanishing coordinate x_{j_P}, $0 \leq j_P \leq n$; we make this choice so that j_P is constant on orbits. Given $\underline{F} \in S_{\underline{d}}(\overline{\mathbb{F}}_q)$, write $\underline{F}_P$ for the image of $(F_1/x_{j_P}^{d_1}, \dots, F_r/x_{j_P}^{d_r})$ in $(\mathscr{O}_{\mathbb{P}^n_{\overline{\mathbb{F}}_q},P}/\mathfrak{m}_P^2)^{\oplus r}$.

4.6.1.4 Define

$$L(a, b, c) = \prod_{j=0}^{c-1} (1 - a^{-(b-j)})$$

which, when $a = q$, is the probability that c randomly chosen vectors in $\mathbb{F}_q^b$ are linearly independent.

Proposition 4.6.1. *Let Y be a smooth, quasi-projective subscheme of $\mathbb{P}^n_{\mathbb{F}_q}$ of dimension m.*

Set $B = \mathbb{P}^n_{\mathbb{F}_q}(\overline{\mathbb{F}}_q)$ and for each $P \in B$, let A_P be the subset of $(\mathcal{O}_{\mathbb{P}^n_{\overline{\mathbb{F}}_q},P}/\mathfrak{m}_P^2)^{\oplus r}$ such that

$$A_P = \begin{cases} \left\{ \underline{g}_P \in (\mathcal{O}_{\mathbb{P}^n_{\overline{\mathbb{F}}_q},P}/\mathfrak{m}_P^2)^{\oplus r} \;\middle|\; \begin{array}{l} \text{for } \overline{g}_i \text{ the image of } g_i \text{ in } \mathcal{O}_{Y_{\overline{\mathbb{F}}_q},P} \text{ and} \\ \mathfrak{m} \text{ the maximal ideal in } \mathcal{O}_{Y_{\overline{\mathbb{F}}_q},P}, \\ \text{not all } \overline{g}_i \text{ lie in } \mathfrak{m} \text{ or all } \overline{g}_i \text{ lie in } \mathfrak{m} \\ \text{and are linearly independent in } \mathfrak{m}/\mathfrak{m}^2 \end{array} \right\} & P \in Y \\ (\mathcal{O}_{\mathbb{P}^n_{\overline{\mathbb{F}}_q},P}/\mathfrak{m}_P^2)^{\oplus r} & P \notin Y \end{cases}$$

where by g_i we mean the image of the i^{th} component of $\underline{g}_P$ in $\mathcal{O}_{Y_{\overline{\mathbb{F}}_q},P}/\mathfrak{m}_P^2$.

Set $A = \bigsqcup_{P \in B} A_P$. Let $U_{\underline{d}}$ be the set of $\underline{F} \in S_{\underline{d}}(\overline{\mathbb{F}}_q)$ such that for all $P \in B$, $\underline{F}_P$ lies in A_P. Let $A \to B$ be the map $\underline{F}_P \mapsto P$ and $\mathrm{ev}_{\underline{d}} : U_{\underline{d}} \times B \to A$ the map $(\underline{F}, P) \mapsto \underline{F}_P$. Then $(U_{\underline{d}}, \mathrm{ev}_{\underline{d}})_{\underline{d} \in I_{m,r}}$ equidistributes on A/B in the sense of Definition 4.4.1.

Proof. For an admissible $\mathbb{Z}$-subset $B' \subseteq B_{\mathbf{k}}$ of finite cardinality, we have

$$\mathrm{Hom}_{B_{\mathbf{k}}}(B', A_{\mathbf{k}}) = \prod_{|P| \in |B'|} \mathrm{Hom}_{B_{\mathbf{k}}}(|P|, A_{\mathbf{k}}).$$

If we identify the orbit $|P| \subseteq B' \subseteq \mathbb{P}^n(\overline{\mathbb{F}}_q)$ with a closed point of $\mathbb{P}^n_{\mathbb{F}_{q^k}}$, then we obtain a canonical identification of $\mathrm{Hom}_{B_{\mathbf{k}}}(|P|, A_{\mathbf{k}})$ with a subset $A_{|P|}$ of $(\mathcal{O}_{\mathbb{P}^n_{\mathbb{F}_{q^k}},|P|}/\mathfrak{m}_{|P|}^2)^{\oplus r}$. Viewing $U_{\underline{d},\mathbf{k}}(\mathbf{1})$ as the set of $\underline{F} \in S_{\underline{d}}(\mathbb{F}_{q^k})$ such that the image of $\underline{F}$ in $(\mathcal{O}_{\mathbb{P}^n_{\mathbb{F}_{q^k}},|P|}/\mathfrak{m}_{|P|}^2)^{\oplus r}$ lies in $A_{|P|}$ for all $|P| \in |\mathbb{P}^n_{\mathbb{F}_{q^k}}|$, the map $\mathrm{ev}_{\underline{d},B'}$ sends $\underline{F}$ to the tuple $(\underline{F}_{|P|})_{|P| \in |B'|}$.

To establish Equation (4.4.1) it suffices to show equality for each singleton $\{(\underline{F}_{|P|})_{|P| \in |B'|}\}$. On the right side we have

$$\begin{aligned} &\mu_{\mathrm{Hom}_{B_{\mathbf{k}}}(B', A_{\mathbf{k}})}(\{(\underline{F}_{|P|})_{|P|}\}) \\ &\quad = \prod_{|P| \in |B'|} \frac{1}{\# A_{|P|}} \\ &\quad = \Bigg(\prod_{|P| \in |B'| - |Y_{\mathbb{F}_{q^k}}|} q^{-kr \deg(P)(n+1)} \Bigg) \Bigg(\prod_{|P| \in |B'| \cap |Y_{\mathbb{F}_{q^k}}|} \frac{q^{-kr \deg(P)(n+1)}}{1 - q^{-kr \deg(P)} + q^{-kr \deg(P)} L(q^{k \deg(P)}, m, r)} \Bigg). \end{aligned}$$

The left side of Equation (4.4.1) can be viewed as a conditional probability:

$$\begin{aligned}
&\big((\mathrm{ev}_{\underline{d},B'})_*\mu_{U_{\underline{d},\mathbf{k}}(\mathbf{1})}\big)(\{(\underline{F}_{|P|})_{|P|}\}) \\
&\quad= \mu_{U_{\underline{d},\mathbf{k}}(\mathbf{1})}(\mathrm{ev}^{-1}_{\underline{d},B'}(\{(\underline{F}_{|P|})_{|P|}\}) \\
&\quad= \frac{\#\{\underline{G}\in S_{\underline{d}}(\mathbb{F}_{q^k}) \mid \underline{G}_{|P|}=\underline{F}_{|P|} \text{ for all } |P|\in|B'| \text{ and } \underline{G}\in U_{\underline{d},\mathbf{k}}(\mathbf{1})\}/\#S_{\underline{d}}(\mathbb{F}_{q^k})}{\#U_{\underline{d},\mathbf{k}}(\mathbf{1})/\#S_{\underline{d}}(\mathbb{F}_{q^k})}.
\end{aligned}$$

By [BK12, Theorem 1.2], as $\underline{d}\to\infty$ such that $\max(\underline{d})^m q^{-\min(\underline{d})/(m+1)}\to 0$ (guaranteed by our choice of $I_{m,r}$), this converges to

$$\begin{aligned}
&\frac{\Big(\prod_{|P|\in|B'|} q^{-kr\deg(P)(n+1)}\Big)\Big(\prod_{|P|\in|Y_{\mathbb{F}_{q^k}}|-|B'|}\big(1-q^{-kr\deg(P)}+q^{-kr\deg(P)}L(q^{k\deg(P)},m,r)\big)\Big)}{\prod_{|P|\in|Y_{\mathbb{F}_{q^k}}|}\big(1-q^{-kr\deg(P)}+q^{-kr\deg(P)}L(q^{k\deg(P)},m,r)\big)} \\
&= \Big(\prod_{|P|\in|B'|-|Y_{\mathbb{F}_{q^k}}|} q^{-kr\deg(P)(n+1)}\Big)\Big(\prod_{|P|\in|B'|\cap|Y_{\mathbb{F}_{q^k}}|}\frac{q^{-kr\deg(P)(n+1)}}{1-q^{-kr\deg(P)}+q^{-kr\deg(P)}L(q^{k\deg(P)},m,r)}\Big).
\end{aligned}$$

So Equation (4.4.1) is satisfied, and thus $(U_{\underline{d}},\mathrm{ev}_{\underline{d}})$ equidistributes on A/B. □

4.6.2 Application: Zeta functions and L-functions of complete intersections

Let κ be a finite field of order q and fix an algebraic closure $\overline{\kappa}$. Let $Y\subseteq\mathbb{P}^n_\kappa$ be a smooth quasi-projective subscheme of dimension $m+r$. With notation as in Proposition 4.6.1, for $F\in U_{\underline{d}}$ we write $C_{\underline{F}}$ for the scheme-theoretic intersection $Y\cap V(F_1)\cap\ldots\cap V(F_r)$, a smooth quasi-projective subscheme of $\mathbb{P}^n_{\kappa(\underline{F})}$, where $\kappa(\underline{F})$ is the subfield of $\overline{\kappa}$ generated by the coefficients of $F_1,\ldots,F_r$.

Let $X_{\underline{d}}$ be the random variable on $U_{\underline{d}}$ sending $\underline{F}$ to

$$Z_{C_{\underline{F}}}(t)=\prod_{y\in|C_{\underline{F}}|}\frac{1}{1-t^{\deg y}}.$$

The following (combined with [How25, Theorem 2.2.1]) generalizes [How24, Theorem 8.3.1], which is the case of $r=1$.

Theorem 4.6.2. *With notation as above, as $\underline{d}$ goes to ∞ in $I_{m+r,r}$ (see 4.6.1.2), the Λ-distribution of $X_{\underline{d}}$ converges to a binomial Λ-distribution with parameters*

$$p=\frac{[q]^{-r}L([q],m+r,r)}{1-[q]^{-r}+[q]^{-r}L([q],m+r,r)} \quad\text{and}\quad N=[Y(\overline{\kappa})],$$

i.e.,

$$\lim_{d\in I_{m+r,r}}\mathbb{E}[\mathrm{Exp}_\sigma(X_{\underline{d}}h_1)]=(1+p(h_1+h_2+\ldots))^N.$$

Proof. We will use the notation of Proposition 4.6.1. For $P \in Y(\overline{\kappa})$, we consider the random variable $\mathcal{X}_P$ on A_P that sends a germ f to

$$\begin{cases} \frac{1}{1-t} & \text{if } f_i(P) = 0 \text{ for all } i \\ 0 & \text{otherwise.} \end{cases}$$

For $P \in \mathbb{P}^n(\overline{\kappa}) - Y(\overline{\kappa})$, we set $\mathcal{X}_P$ to be the trivial random variable. Then

$$X_d = \int_{U_d \times \mathbb{P}^n_{\overline{\kappa}}/U_d} \mathrm{ev}_d^* \mathcal{X}.$$

By Proposition 4.6.1, $(U_{\underline{d}}, \mathrm{ev}_{\underline{d}})$ equidistributes on $A/\mathbb{P}^n(\overline{\kappa})$, thus, by Theorem 4.4.3, the asymptotic σ-moment-generating function is

$$\prod_{\mathbb{P}^n(\overline{\kappa})} \mathbb{E}_{A/\mathbb{P}^n(\overline{\kappa})}[\mathrm{Exp}_\sigma(\mathcal{X} h_1)] = \prod_{Y(\overline{\kappa})} \mathbb{E}_{A_{Y(\overline{\kappa})}/Y(\overline{\kappa})}[\mathrm{Exp}_\sigma(\mathcal{X} h_1)]$$

where the equality is because the moment-generating function of the trivial random variable over $\mathbb{P}^n(\overline{\kappa}) - Y(\overline{\kappa})$ is identically 1.

For $P \in Y(\overline{\kappa})$ of degree k, $\mathcal{X}_P$ is a Bernoulli random variable equal to $\frac{1}{1-t}$ (the unit in $W(\mathbb{C})$) with probability

$$\frac{[q^k]^{-r} L([q^k], m+r, r)}{1 - [q^k]^{-r} + [q^k]^{-r} L([q^k], m+r, r)}.$$

Indeed, this can be checked on each ghost component using Lemma 4.2.3; the numerator gives the probability that an r-tuple of germs vanishing at a point are transverse at that point, while the denominator gives the probability that an r-tuple of germs either do not all vanish at a point or all vanish and are transverse.

It follows that $\mathrm{Exp}_\sigma(\mathcal{X} h_1)|_{Y(\overline{\kappa})}$ is the pullback of $(1 + p(h_1 + h_2 + \ldots))$ from $\mathbf{1}$ to $Y(\overline{\kappa})$. Thus, using Example 4.3.3 for the second equality,

$$\prod_{Y(\overline{\kappa})} \mathbb{E}_{A_{Y(\overline{\kappa})}/Y(\overline{\kappa})}[\mathrm{Exp}_\sigma(\mathcal{X} h_1)] = \prod_{Y(\overline{\kappa})} (1 + p(h_1 + h_2 + \ldots)) = (1 + p(h_1 + h_2 + \ldots))^{[Y(\overline{\kappa})]}.$$

□

4.6.2.1 We now prove Theorem D. We continue with the notation above, except we now take Y to be a smooth, *closed, and geometrically connected* subscheme in order to agree with the setup in Section 4.1.2 (these conditions show up in the definition of vanishing cohomology and in the analysis of the top degree cohomology when establishing congruences modulo $W(\mathbb{C})^{\mathrm{bdd}}$).

Proof of Theorem D. We write $X_{\underline{d}}$ for the random variable on $U_{\underline{d}}$ as in Section 4.1.2 sending $\underline{F}$ to $\mathcal{L}_{C_{\underline{F}}}(t)$. For $X_{\underline{d}}$ as above, we have, as in the $r = 1$ case of [How24, §8.4],

$$X_{\underline{d}} = [q^{-m/2}]\left((-1)^m X_{\underline{d}} - [H^m(Y)] - (-1)^m \sum_{i=0}^{m-1}(-1)^i(1+[q^{m-i}])[H^i(Y)]\right). \tag{4.6.1}$$

Because any constant random variable is independent to any other random variable, we find

$$\mathbb{E}[\mathrm{Exp}_\sigma(X_{\underline{d}}h_1)] = \mathbb{E}[\mathrm{Exp}_\sigma((-1)^m[q^{-m/2}]X_{\underline{d}})]\,\mathrm{Exp}_\sigma(\mu h_1)$$

for

$$\mu = -[q^{-m/2}][H^m(Y)] - (-1)^m \sum_{i=0}^{m-1}(-1)^i([q^{-m/2}]+[q^{m/2-i}])[H^i(Y)].$$

To obtain Equation (4.1.1), it remains to note that, by Theorem 4.6.2,

$$\lim_{\underline{d}\in I_{m+r,r}} \mathbb{E}\left[\mathrm{Exp}_\sigma([q^{-m/2}]X_{\underline{d}})\right] = (1+p([q^{-m/2}]h_1 + [q^{-m}]h_2 + \ldots))^{[Y(\overline{\kappa})]}$$

and thus also, by [How25, Theorem 2.2.1],

$$\lim_{\underline{d}\in I_{m+r,r}} \mathbb{E}\left[\mathrm{Exp}_\sigma(-[q^{-m/2}]X_{\underline{d}})\right] = (1+p(-[q^{-m/2}]e_1 + [q^{-m}]e_2 - \ldots))^{[Y(\overline{\kappa})]}.$$

It remains just to establish the claimed comparisons mod $[q^{-1/2}]W(\mathbb{C})^{\mathrm{bdd}}$. This is nearly identical to the proof of [How24, Proposition 9.2.2] after we establish

$$p \equiv [q^{-r}] \mod [q^{-(m+1+r)}]W(\mathbb{C})^{\mathrm{bdd}}.$$

But this is immediate if we note $L([q], m+r, r) = \prod_{j=0}^{r-1}(1-[q]^{-(m+r-j)})$ then expand

$$p = \frac{[q]^{-r}L([q],m+r,r)}{1-[q]^{-r}+[q]^{-r}L([q],m+r,r)} = \frac{[q^{-r}]-[q^{-(m+1)-r}]+\ldots}{1-[q^{-(m+1)-r}]+\ldots}.$$

□

4.7 Semiample equidistribution

In this section, we first establish an equidistribution result for sections of semiample bundles using the generalization of Poonen's sieve in [EW15], Proposition 4.7.1. We then combine Proposition 4.7.1 with Theorem 4.4.3 to compute, in Theorem 4.7.2, the asymptotic Λ-distribution of the zeta functions of curves of bidegree $(2, d)$ on Hirzebruch surfaces (generalizing the computation of the classical distribution of rational points given in [EW15, Theorem 9.9-(b)]).

4.7.1 Establishing equidistribution

Let Y be a smooth, projective scheme (integral but not necessarily geometrically integral) of dimension m over $\mathbb{F}_q$ with q a power of a prime p. Consider a very ample divisor D on Y and a globally generated divisor E on Y. Let π be the map given by the complete linear series on E:

$$\pi : Y \xrightarrow{|E|} \mathbb{P}^M_{\mathbb{F}_q}.$$

Define $R_{n,d} := H^0(Y, \mathcal{O}_Y(nD + dE))$ and for $F \in R_{n,d}$, write H_F for the corresponding divisor in $|nD + dE|$.

Suppose $z \in |\pi(Y)| \subseteq |\mathbb{P}^M_{\mathbb{F}_q}|$ is a closed point and $z^{(1)} := \operatorname{Spec}(\mathcal{O}_{\mathbb{P}^M_{\mathbb{F}_q},z}/\mathfrak{m}_z^2)$ the first-order infinitesimal neighborhood of z. For $y \in |Y|$ a closed point in $\pi^{-1}(z)$, let $y^{(1)} = \operatorname{Spec}(\mathcal{O}_{Y,y}/\mathfrak{m}_y^2)$. For any finite subscheme $W \subset \mathbb{P}^M_{\mathbb{F}_q}$, define $Y_W = Y \times_{\mathbb{P}^M_{\mathbb{F}_q}} W$.

Given a section $F \in R_{n,d}$, H_F is smooth at a closed point $y \in \pi^{-1}(z)$ if and only if F does not vanish under the restriction map

$$R_{n,d} \longrightarrow H^0(y^{(1)}, \mathcal{O}_{y^{(1)}}(nD)) \cong \mathcal{O}_{Y,y}/\mathfrak{m}_y^2.$$

where the latter isomorphism depends on a choice of trivialization of $\mathcal{O}_{y^{(1)}}(nD)$, but the condition of non-vanishing does not. This restriction map factors as

$$R_{n,d} \xrightarrow{\alpha} H^0(Y_{y^{(1)}}, \mathcal{O}_{Y_{y^{(1)}}}(nD)) \longrightarrow \mathcal{O}_{Y,y}/\mathfrak{m}_y^2.$$

Set $\mathscr{F} = \pi_*(\mathcal{O}_Y(nD))$, so $\mathscr{F}(d) \cong \pi_*(\mathcal{O}_Y(nD + dE))$.

There is a natural map

$$\mathscr{F}(d) \otimes_{\mathcal{O}_{\mathbb{P}^M}} \mathcal{O}_{z^{(1)}} \xrightarrow{\beta} H^0\big(Y_{y^{(1)}}, \mathcal{O}_{Y_{y^{(1)}}}(nD)\big).$$

Note that α is the composition of β and the natural restriction map

$$R_{n,d} = H^0(\mathbb{P}^M_{\mathbb{F}_q}, \pi_*\mathcal{O}_Y(nD + dE)) = H^0(\mathbb{P}^M_{\mathbb{F}_q}, \mathscr{F}(d)) \longrightarrow \mathscr{F}(d) \otimes_{\mathcal{O}_{\mathbb{P}^M}} \mathcal{O}_{z^{(1)}}. \tag{4.7.1}$$

As explained in the proof of [EW15, Lemma 5.2(a)], Serre vanishing implies that for $d \gg 0$ this restriction map Equation (4.7.1) is surjective.

Let $\phi : \mathbb{P}^M_{\overline{\mathbb{F}}_q} \to \mathbb{P}^M_{\mathbb{F}_q}$ (resp. $\phi_k : \mathbb{P}^M_{\overline{\mathbb{F}}_q} \to \mathbb{P}^M_{\mathbb{F}_{q^k}}$) be the natural map. Let $\pi' : Y_{\overline{\mathbb{F}}_q} \to \mathbb{P}^M_{\overline{\mathbb{F}}_q}$ be the base change of π relative to ϕ. We write the homogeneous coordinates on $\mathbb{P}^M_{\mathbb{F}_q}$ as $x_0, \ldots, x_M$, and for each $P \in (\pi(Y))(\overline{\mathbb{F}}_q)$, we fix a $0 \le j_P \le M$ such that x_{j_P} does not vanish at P; we make this choice so that j_P is constant on orbits. For $F \in R_{n,d}(\overline{\mathbb{F}}_q)$, we write F_P for the image of $F/x_{j_P}^d$ in $\phi^*\mathscr{F} \otimes_{\mathcal{O}_{\mathbb{P}^M_{\overline{\mathbb{F}}_q}}} \mathcal{O}_{P^{(1)}}$.

Proposition 4.7.1. *With notation as above, set* $B = (\pi(Y))(\overline{\mathbb{F}}_q)$. *For each* $P \in B$, *let* A_P *be the set of* $g_P \in \phi^*\mathscr{F} \otimes_{\mathscr{O}_{\mathbb{P}^M_{\mathbb{F}_q}}} \mathscr{O}_{P^{(1)}}$ *such that the image of* g_P *in* $\mathscr{O}_{Q^{(1)}}$ *is nonzero for all* $Q \in \pi'^{-1}(P)$.

Set $A = \bigsqcup_{P \in B} A_P$. *Let* U_d *be the set of* $F \in R_{n,d}(\overline{\mathbb{F}}_q)$ *such that for all* $P \in B$, *the image of* F *in* $\mathscr{O}_{Q^{(1)}}$ *is nonzero for all* $Q \in \pi'^{-1}(P)$. *Each of* A, B, *and* U_d *are admissible* $\mathbb{Z}$*-sets with the geometric Frobenius action.*

Let $A \to B$ *be the map* $g_P \mapsto P$ *and* $\mathrm{ev}_d : U_d \times B \to A$ *the map* $(F, P) \mapsto F_P$. *If* $n \geq \max\{(\dim \pi(Y))(m+1) - 1, (\dim \pi(Y))p + 1\}$, *then* (U_d, ev_d) *equidistributes on* A/B *in the sense of Definition 4.4.1.*

Proof. For an admissible $\mathbb{Z}$-subset $B' \subseteq B_{\mathbf{k}}$ of finite cardinality, we have

$$\mathrm{Hom}_{B_{\mathbf{k}}}(B', A_{\mathbf{k}}) = \prod_{|P| \in |B'|} \mathrm{Hom}_{B_{\mathbf{k}}}(|P|, A_{\mathbf{k}}).$$

If we identify the orbit $|P| \subseteq B' \subseteq \mathbb{P}^M(\overline{\mathbb{F}}_q)$ with a closed point of $\mathbb{P}^M_{\mathbb{F}_{q^k}}$, then we obtain a canonical identification of $\mathrm{Hom}_{B_{\mathbf{k}}}(|P|, A_{\mathbf{k}})$ with a subset $A_{|P|}$ of $\mathscr{F} \otimes_{\mathscr{O}_{\mathbb{P}^M_{\mathbb{F}_{q^k}}}} \mathscr{O}_{|P|^{(1)}}$. Viewing $U_{d,\mathbf{k}}(\mathbf{1})$ as the set of $F \in R_{n,d}(\mathbb{F}_{q^k})$ such that the image $F_{|P|}$ of F in $\mathscr{F} \otimes_{\mathscr{O}_{\mathbb{P}^M_{\mathbb{F}_{q^k}}}} \mathscr{O}_{|P|^{(1)}}$ lies in $A_{|P|}$ for all $|P| \in |B_{\mathbf{k}}| \subseteq |\mathbb{P}^n_{\mathbb{F}_{q^k}}|$, the map $\mathrm{ev}_{d,B'}$ sends F to the tuple $(F_{|P|})_{|P| \in |B'|}$.

To establish Equation (4.4.1), it suffices to show equality for each singleton $\{(F_{|P|})_{|P| \in |B'|}\}$. On the right side we have

$$\mu_{\mathrm{Hom}_{B_{\mathbf{k}}}(B', A_{\mathbf{k}})}(\{(F_{|P|})_{|P|}\}) = \prod_{|P| \in |B'|} \frac{1}{\# A_{|P|}}.$$

The left side can be viewed as a conditional probability:

$$\begin{aligned}
&\big((\mathrm{ev}_{d,B'})_* \mu_{U_{d,\mathbf{k}}(\mathbf{1})}\big)(\{(F_{|P|})_{|P|}\}) \\
&\quad = \mu_{U_{d,\mathbf{k}}(\mathbf{1})}(\mathrm{ev}^{-1}_{d,B'}(\{(F_{|P|})_{|P|}\}) \\
&\quad = \frac{\#\{G \in R_{n,d}(\mathbb{F}_{q^k}) \mid G_{|P|} = F_{|P|} \text{ for all } |P| \in |B'| \text{ and } G \in U_{d,\mathbf{k}}(\mathbf{1})\} / \# R_{n,d}(\mathbb{F}_{q^k})}{\# U_{d,\mathbf{k}}(\mathbf{1}) / \# R_{n,d}(\mathbb{F}_{q^k})}.
\end{aligned}$$

Since the restriction maps of Equation (4.7.1) (or rather their analogs over $\mathbb{F}_{q^k}$) are surjective for $d \gg 0$, we can phrase local probabilities at $|P|$ in terms of $\phi_k^* \mathscr{F}(d) \otimes_{\mathscr{O}_{\mathbb{P}^M_{\mathbb{F}_{q^k}}}} \mathscr{O}_{|P|^{(1)}}$ instead of $R_{n,d}$. Thus, by [EW15, Theorem 3.1], as $d \to \infty$, this converges to

$$\frac{\left(\prod_{|P| \in |B'|} \frac{1}{\# \phi_k^* \mathscr{F} \otimes_{\mathscr{O}_{\mathbb{P}^M_{\mathbb{F}_{q^k}}}} \mathscr{O}_{|P|^{(1)}}}\right)\left(\prod_{|P| \in |\pi(Y)_{\mathbb{F}_{q^k}}| - |B'|} \frac{\# A_{|P|}}{\# \phi_k^* \mathscr{F} \otimes_{\mathscr{O}_{\mathbb{P}^M_{\mathbb{F}_{q^k}}}} \mathscr{O}_{|P|^{(1)}}}\right)}{\prod_{|P| \in |\pi(Y)_{\mathbb{F}_{q^k}}|} \frac{\# A_{|P|}}{\# \phi_k^* \mathscr{F} \otimes_{\mathscr{O}_{\mathbb{P}^M_{\mathbb{F}_{q^k}}}} \mathscr{O}_{|P|^{(1)}}}} = \prod_{|P| \in |B'|} \frac{1}{\# A_{|P|}}.$$

So Equation (4.4.1) is satisfied, and thus (U_d, ev_d) equidistributes on A/B. □

4.7.2 Application: Zeta functions of curves on Hirzebruch surfaces

In [EW15, Theorem 9.9], the semiample Bertini theorem is used to compute the asymptotic distribution of some point-counting random variables for smooth curves on Hirzebruch surfaces. Using our methods, this can be extended to compute the full Λ-distributions — the classical distributions in [EW15, Theorem 9.9] are equivalent to the restriction of the Λ-distributions to $\mathbb{Z}[h_1] \subseteq \Lambda$. We illustrate this below in the case of bidegree $(2, d)$ curves on Hirzebruch surfaces ([EW15, Theorem 9.9-(b)]).

4.7.2.1 Let κ be a finite field of order q, let $Y = \mathrm{Proj}_{\mathbb{P}^1_\kappa}(\mathrm{Sym}^\bullet(\mathcal{O} \oplus \mathcal{O}(a)))$ for $a \geq 0$ be a Hirzebruch surface over κ, and write $\pi : Y \to \mathbb{P}^1_\kappa$ for its natural projection. We let E be the divisor on Y of the fiber over ∞ on $\mathbb{P}^1_\kappa$ (so that π is induced by E) and let D the class of a hyperplane section in the relative proj construction. We write $\mathcal{O}(i, j) = \mathcal{O}(iD + jE)$, a line bundle on Y.

Let U_d be the admissible $\mathbb{Z}$-set of global sections of $\mathcal{O}(2, d)$ on $Y_{\overline{\kappa}}$ with smooth vanishing locus. For F in U_d, we write $\kappa(F)$ for the subfield of $\overline{\kappa}$ generated by the coefficients of F and κ, and we view the vanishing locus $V(F)$ as a scheme over $\kappa(F)$.

Let X_d be the random variable on U_d sending F to the zeta function

$$Z_{V(F)}(t) = \prod_{y \in |V(F)|} \frac{1}{1 - t^{\deg y}} = \prod_{z \in |\mathbb{P}^1_{\kappa(F)}|} Z_{\pi^{-1}(z) \cap V(F)}(t).$$

Theorem 4.7.2. *With notation as above,*

$$\lim_{d \to \infty} \mathbb{E}[\mathrm{Exp}_\sigma(X_d h_1)]$$
$$= \left(\frac{([q]^2 - 1)([q] - 1)\left(\sum_{j \geq 0} h_j\right) + \frac{[q]^4 - [q]^2}{2}\left(\sum_{j \geq 0} h_j\right)^2 + \frac{([q]^2 - [q])^2}{2}\left(\sum_{j \geq 0} h_j(\underline{t}^2)\right)}{[q]^4 - [q]^2 - [q] + 1} \right)^{\mathbb{P}^1(\overline{\kappa})}.$$

Proof. We adopt the notation of Proposition 4.7.1 for our choice of Y, D, and E above. We note that, although $n = 2$ does not satisfy the bounds given in Proposition 4.7.1, by [EW15, Proposition 8.2], the application of [EW15, Theorem 3.1] in the proof of Proposition 4.7.1 is still valid in this specific setting, so that we have equidistribution.

For $P \in \mathbb{P}^1(\overline{\kappa})$, let $\mathcal{X}_P$ be the random variable on A_P that sends a germ g_P to $Z_{V(\overline{g}_P)}(t)$, where here the vanishing locus is taken inside of $Y_P \cong \mathbb{P}^1_{\overline{\kappa}}$, $\overline{g}_P$ is the induced element of $H^0(Y_P, \mathcal{O}(2D)) \cong H^0(\mathbb{P}^1_{\overline{\kappa}}, \mathcal{O}(2))$, and we treat $V(\overline{g}_P)$ as being defined over $\kappa(g_P)$ to obtain a zeta function.

We then have

$$X_d = \int_{U_d \times \mathbb{P}^1(\overline{\kappa})/U_d} \mathrm{ev}_d^* \mathcal{X}$$

and thus, by Theorem 4.4.3,

$$\lim_{d\to\infty} \mathbb{E}[\mathrm{Exp}_\sigma(X_d h_1)] = \prod_{\mathbb{P}^1(\overline{\kappa})} \mathbb{E}_{A/\mathbb{P}^1(\overline{\kappa})}[\mathrm{Exp}_\sigma(\mathcal{X} h_1)].$$

We now compute the σ-moment-generating function for $\mathcal{X}_P$, $P \in \mathbb{P}^1(\overline{\kappa})$, using Lemma 4.2.3. Let $q_P = \#\kappa(P)$. We note that $\mathrm{res}_k(\mathcal{X}_P)$ can be viewed as a function on the germs at P defined over $\kappa(P)_k$, the degree k extension of $\kappa(P)$ in $\overline{\kappa}$. On such a germ g, it takes value

(a) $\frac{1}{1-t}$ if $\overline{g}$ is the square of a single factor. The number of such cases is

$$(q_P^{2k}-1)\cdot(q_P^{3k}-q_P^{2k}).$$

Here the $q_P^{2k}-1=(q_P^k+1)(q_P^k-1)$ is the number of points in $\mathbb{P}^1(\kappa(P)_k)$ times the number of degree two homogeneous equations vanishing at such a point with multiplicity two, and the factor $q_P^{3k}-q_P^{2k}$ is the number of possible smooth extensions g of each $\overline{g}$ (which correspond to degree two polynomials that don't have a zero at the same point — cf. [EW15, Lemma 9.8 and preceding paragraph])[1].

(b) $\left(\frac{1}{1-t}\right)^2$ if $\overline{g}$ splits into two distinct factors over $\kappa(P)_k$. The number of such cases is

$$\frac{(q_P^k+1)q_P^k}{2}(q_P^k-1)q_P^{3k} = \frac{q_P^{2k}-1}{2}q_P^{4k}$$

where $\frac{(q_P^k+1)q_P^k}{2}(q_P^k-1)$ is the number of pairs of distinct points in $\mathbb{P}^1(\kappa(P)_k)$ times the number of degree two homogeneous equations vanishing exactly at such a pair, and q_P^{3k} counts the number of smooth extensions g of each $\overline{g}$ (which correspond to arbitrary degree 3 polynomials).

(c) $\frac{1}{1-t^2}$ if $\overline{g}$ is irreducible over $\kappa(P)_k$. The number of such cases is

$$\frac{q_P^{2k}-q_P^k}{2}(q_P^k-1)q_P^{3k} = \frac{(q_P^k-1)^2}{2}q_P^{4k}$$

where $\frac{q_P^{2k}-q_P^k}{2}(q_P^k-1)$ is the number of degree 2 closed points in $\mathbb{P}^1_{\kappa(P)_k}$ times the number of degree two homogeneous equations vanishing exactly at such a point, and q_P^{3k} counts the number of smooth extensions g of each $\overline{g}$ (which correspond to arbitrary degree 3 polynomials).

[1] One can also compare this computation with [EW15, proof of Proposition 9.9-(b)], but note that there is a typo in the corresponding computation in that proof: the second $(q-1)(q+1)$ appearing should in fact be our q^3-q^2.

Note that the total number of cases adds up to $q_P^{6k} - q_P^{4k} - q_P^{3k} + q_P^{2k}$, i.e. this is the denominator for the probability of each case occurring.

Now we note that

$$\mathrm{Exp}_\sigma\left(\frac{1}{1-t}h_1\right) = \sum_{j\geq 0} h_j$$

and

$$\mathrm{Exp}_\sigma\left(\left(\frac{1}{1-t}\right)^2 h_1\right) = \left(\mathrm{Exp}_\sigma\left(\frac{1}{1-t}h_1\right)\right)^2 = \left(\sum_{j\geq 0} h_j\right)^2.$$

The formula for $\mathrm{Exp}_\sigma(\frac{1}{1-t^2})$ is more complicated, but we only need the first component, which is straightforward:

$$\mathrm{Exp}_\sigma\left(\frac{1}{1-t^2}h_1\right)_1 = \sum_\tau \left(h_\tau \circ \frac{1}{1-t^2}\right)_1 m_\tau = \sum_\tau m_{2\tau} = \sum_\tau m_\tau(\underline{t}^2) = \sum_j h_j(\underline{t}^2).$$

The first equality is [How24, Example 2.5.2] and the second follows because $\frac{1}{1-t^2} = [\mathbf{2}]$ so that $h_j \circ \frac{1}{1-t^2} = [\mathrm{Sym}^j(\mathbf{2})]$; indeed, $\mathrm{Sym}^j(\mathbf{2})$ has one fixed point if j is even and no fixed points otherwise. Combining our computation above of the probability of each value $\bullet$ with these computations of $\mathrm{Exp}_\sigma(\bullet)_1$, we obtain

$$\begin{aligned}&\mathbb{E}_1[\mathrm{Exp}_\sigma(\mathrm{res}_k(\mathcal{X}_P h_1))]\\ &\quad= \frac{(q_P^{2k}-1)(q_P^k-1)\left(\sum_{j\geq 0} h_j\right) + \frac{q_P^{4k}-q_P^{2k}}{2}\left(\sum_{j\geq 0} h_j\right)^2 + \frac{(q_P^{2k}-q_P^k)^2}{2}\left(\sum_{j\geq 0} h_j(\underline{t}^2)\right)}{q_P^{4k}-q_P^{2k}-q_P^k+1}.\end{aligned}$$

Applying Lemma 4.2.3, and comparing k^{th} components, we find

$$\begin{aligned}&\mathbb{E}[\mathrm{Exp}_\sigma(\mathcal{X}_P h_1)]\\ &\quad= \frac{([q_P]^2-1)([q_P]-1)\left(\sum_{j\geq 0} h_j\right) + \frac{[q_P]^4-[q_P]^2}{2}\left(\sum_{j\geq 0} h_j\right)^2 + \frac{([q_P]^2-[q_P])^2}{2}\left(\sum_{j\geq 0} h_j(\underline{t}^2)\right)}{[q_P]^4-[q_P]^2-[q_P]+1}.\end{aligned}$$

The function on $\mathbb{P}^1(\overline{\kappa})$ sending P to this series is the pullback from $\mathbf{1}$ of

$$\frac{([q]^2-1)([q]-1)\left(\sum_{j\geq 0} h_j\right) + \frac{[q]^4-[q]^2}{2}\left(\sum_{j\geq 0} h_j\right)^2 + \frac{([q]^2-[q])^2}{2}\left(\sum_{j\geq 0} h_j(\underline{t}^2)\right)}{[q]^4-[q]^2-[q]+1}.$$

Thus, applying Example 4.3.3 to compute $\prod_{\mathbb{P}^1(\overline{\kappa})} \mathbb{E}_{A/\mathbb{P}^1(\overline{\kappa})}[\mathrm{Exp}_\sigma(\mathcal{X}h_1)]$, we obtain the claimed expression. □

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