1. The Hilbert Scheme. Questions in algebraic geometry of the form:

"How many objects with property X are there?"

are interpreted in terms of intersection theory on moduli spaces. Algebraic geometers have been doing this implicitly for more than 150 years, but the tools to make the arguments rigorous are of a much more recent vintage. In the early 1960's, Grothendieck introduced a category-theoretic framework for considering moduli problems and constructed the Hilbert scheme, which is certainly the most fundamental of the moduli spaces. This is a non-trivial construction, and setting up the background in algebraic geometry will now occupy us for some time.

Schemes: Unlike algebraic varieties, schemes are constructed in such a way that closed subschemes "remember their equations." This leads to many useful properties, which explains the near-universal adoption of schemes throughout algebraic geometry. Here is a brief introduction.

A **locally ringed space** is a pair $(X, \mathcal{O}_X)$ consisting of a topological space X together with a sheaf of rings $\mathcal{O}_X$ with the property that all the stalks $\mathcal{O}_{X,x}$ are local rings (with maximal ideal written as m_x). A morphism of locally ringed spaces is also a pair:

$$f : X \to Y \quad \text{and} \quad f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$$

such that f is continuous and $f^\#$ induces a local homomorphism on stalks, i.e. $f_x^\# : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ satisfies ${f^\#}^{-1}(m_x) = m_{f(x)}$.

Example. A differentiable (resp. complex) manifold with its sheaf of $\mathcal{C}^\infty$ (resp. holomorphic) functions is a locally ringed space, since the rings of germs of such functions are local. A differentiable (resp. holomorphic) map determines a morphism with $f^\# = f^*$, the pull-back map on functions.

Affine schemes are the analogues of open subsets of $\mathbf{R}^n$. Let A be a commutative ring with 1. Then:

$\mathrm{Spec}(A) = \{\text{prime ideals in } A\}$ with the Zariski topology. In particular,

$U_f = \{\text{prime ideals not containing } f\}$ are a basis for the topology.

$\widetilde{A}$ is the "natural" sheaf of rings on $\mathrm{Spec}(A)$ satisfying $\widetilde{A}(U_f) \cong A_f$ and $\widetilde{A}_x \cong A_{\mathcal{P}_x}$ (where $\mathcal{P}_x \subset A$ is the prime ideal corresponding to $x \in \mathrm{Spec}(A)$).

Given a ring homomorphism $\phi : B \to A$, the pair:

$f : \mathrm{Spec}(A) \to \mathrm{Spec}(B)$ defined by $\mathcal{P}_{f(x)} = \phi^{-1}(\mathcal{P}_x)$ which satisfies

$f^{-1}(U_g) = U_{\phi(g)}$ for all basic open sets, and

$f^{\#} : \widetilde{B} \to f_*\widetilde{A}$ deduced from the localized maps $\phi_g : B_g \to A_{\phi(g)}$

is a morphism of locally ringed spaces and the contravariant functor:

$$A \mapsto (\mathrm{Spec}(A), \widetilde{A}); \quad \phi \mapsto (f, f^{\#})$$

is fully faithful. Moreover, within the category of locally ringed spaces,

$$(U_f, \widetilde{A}|_{U_f}) \text{ is isomorphic to } (\mathrm{Spec}(A_f), \widetilde{A}_f)$$

so the basic open sets "are" themselves affine schemes.

A **scheme** is a locally ringed space that is locally affine (e.g. an open subset of an affine scheme). Fix a scheme S. Then an **S-scheme** is a scheme X together with a morphism $\pi : X \to S$. A morphism in the category of S-schemes is a map $f : X \to X'$ commuting with the given maps to S:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\pi"'] & & X' \arrow[dl, "\pi'"] \\ & S & \end{tikzcd}$$

It is of great importance that **products exist** in the category of S-schemes.

Projective space over S, denoted $\mathbf{P}^n_S$, is patched from projective spaces over the open affines $\mathrm{Spec}(A) \subset S$ as follows.

$\mathbf{P}^n_A = \{$homogeneous prime ideals in $A[x_0, ..., x_n]$ not containing $\langle x_0, ..., x_n\rangle\}$ with the Zariski topology. In particular, the

$U_f = \{x \in \mathbf{P}^n_A \mid f \notin \mathcal{P}_x\}$ for homogeneous f are a basis for the topology.

$\mathcal{O}_{\mathbf{P}^n_A}$ is the "natural" sheaf of rings on $\mathbf{P}^n_A$ with the property that for all homogeneous $f \in A[x_0, ..., x_n]$ of positive degree, $\mathcal{O}_{\mathbf{P}^n_A}(U_f) \cong A[x_0, ..., x_n]_{(f)}$ (the degree zero part of $A[x_0, ..., x_n]_f$) and for all $x \in \mathbf{P}^n_A$, the stalks satisfy $\mathcal{O}_{\mathbf{P}^n_A, x} \cong A[x_0, ..., x_n]_{(\mathcal{P}_x)}$ (defined similarly).

The fact that $\mathbf{P}^n_A$ (hence $\mathbf{P}^n_S$) is a scheme follows from isomorphisms $(U_f, \mathcal{O}_{\mathbf{P}^n_A}|_{U_f}) \cong (\mathrm{Spec}(A[x_0, ..., x_n]_{(f)}), \widetilde{A[x_0, ..., x_n]_{(f)}})$ for positive degree f's. $\mathbf{P}^n_S$ is an S-scheme via the canonical maps $\mathbf{P}^n_A \to \mathrm{Spec}(A)$.

A morphism of schemes $f : X \to Y$ is a **closed immersion** if f induces a homeomorphism onto a closed subset $Z \subset Y$ (with the induced topology) and $f^\#$ is surjective. **Closed subschemes** of Y are equivalence classes of closed immersions modulo isomorphisms. The closed subschemes of $\mathrm{Spec}(A)$ correspond to ideals $I \subset A$, and it is in this sense that closed subschemes remember their equations. An S-scheme X is **projective** over S if it is isomorphic to a closed subscheme of some $\mathbf{P}^n_S$.

Suppose $S = \mathrm{Spec}(k)$ where k is a field. An S-scheme (or "k-scheme") X is a **variety** if:

(a) X is covered by finitely many affine open sets $\mathrm{Spec}(A_i)$ with the property that each A_i is a finitely generated k-algebra.

(b) X is an irreducible topological space.

(c) $\mathcal{O}_X$ contains no nilpotent elements (i.e. X is reduced).

(d) $(X, \mathcal{O}_X)$ is "separated" (the analogue of Hausdorff), i.e. the diagonal

$$\Delta : X \to X \times X \text{ is a closed immersion}$$

It is a quasi-projective variety if, moreover, X is isomorphic to an open subset of a projective k-scheme (in which case conditions (a) and (d) are redundant).

If we limit ourselves to the case where k is algebraically closed, then Hilbert's Nullstellensatz can be used to link k-varieties to "classical" algebraic geometry via the equivalence of categories:

$$k\text{-varieties (as schemes)} \;\leftrightarrow\; \text{classical abstract varieties over } k$$

where a scheme X is sent to the subset of closed points and the sheaf $\mathcal{O}_X$ is sent to the sheaf of regular functions.

The **Zariski tangent space** T_x to a scheme X at a point $x \in X$ is the vector space $(m_x/m_x^2)^*$ over the residue field $k(x) = \mathcal{O}_{X,x}/m_x$. A variety X over an algebraically closed field k is **non-singular of dimension** d at x if $d = \dim(T_x) = \dim(\mathcal{O}_{X,x})$. X is nonsingular if it is nonsingular at all points.

If we further limit ourselves to the case $k = \mathbf{C}$, the field of complex numbers, then Chow's lemma (generalized by Serre in GAGA) links nonsingular projective varieties to complex manifolds via an equivalence of categories:

$$\text{nonsingular projective } \mathbf{C}\text{-varieties} \;\leftrightarrow\; \text{complex projective manifolds}$$

This completes (for now) our brief introduction to schemes.

Fine Moduli Spaces: If $\mathcal{C}$ is any category, and *Sets* is the category of sets, then an object X in $\mathcal{C}$ determines the contravariant "functor of points" h_X from $\mathcal{C}$ to *Sets* as follows:

(i) For any object S in $\mathcal{C}$,

$$h_X(S) = \{\text{morphisms } \phi : S \to X\}$$

(ii) For any morphism $f : S \to T$ in $\mathcal{C}$,

$$h_X(f) = f^* : h_X(T) \to h_X(S)$$

That is, each $\psi : T \to X$ is taken to $f^*\psi := \psi \circ f : S \to X$.

Ex 1.1. Check that h_X is indeed a functor.

We need one more level of abstraction. The (contravariant) functors from one category to another form a category. Let's denote by $\mathcal{C}/Sets$ the category of contravariant functors from $\mathcal{C}$ to *Sets*. Then the functor of points should be viewed as a functor $h : \mathcal{C} \to \mathcal{C}/Sets$ via:

$$h(X) = h_X \quad \text{and} \quad h(g : X \to Y) = g_* : h_X \to h_Y$$

That is, $g_*(\phi) = g \circ \phi$ and $(g_* \circ f^*) = f^*$, i.e. $g_*(f^*(\psi)) = f^*(g_*(\psi)) = g \circ \psi \circ f$.

Ex 1.2. (a) What are the morphisms in $\mathcal{C}/Sets$?

(b) Check that h is indeed a (covariant) functor.

Yoneda Lemma: h is fully faithful, i.e. for every pair X, Y of objects of $\mathcal{C}$, the map on morphisms is a bijection: $\{f : X \to Y\} \stackrel{h}{\leftrightarrow} \{F : h_X \to h_Y\}$

Ex 1.3. Prove this.

Corollary: If h_X is isomorphic to h_Y, then X is isomorphic to Y.

Proof: The bijection of the Lemma carries one isomorphism to the other.

Example. The "classical" geometric categories of topological spaces and manifolds have a unique (up to isomorphism) one-point object x which has the property that the morphisms $f : x \to X$ correspond to the points of X, giving rise to a natural bijection $h_X(x) = \{f : x \to X\} \leftrightarrow \{x \in X\}$ hence the terminology "functor of points."

Although the category of k-schemes resembles the categories of manifolds in many ways, the morphisms from $\mathrm{Spec}(k)$ to a k-scheme X only pick up the set of "k-rational" points. Even when k is algebraically closed, these only constitute the points corresponding to maximal ideals. Moreover, there are plenty of non-isomorphic one-point k-schemes, including the affine schemes $\mathrm{Spec}(A)$ where A is a finitely generated k-algebra which is a local Artinian ring, i.e. A is of the form:

$$k[x_1, ..., x_n]/\langle x_1, ..., x_n\rangle^m \to A \to 0$$

The "ring of dual numbers" is the simplest non-trivial example:

$$k[\epsilon] \cong k[x]/\langle x^2\rangle$$

An S-scheme X is said to **represent** a functor F from the category of S-schemes to the category of sets if F is isomorphic to h_X. In that case, X is called a **fine moduli space** for the moduli problem given by F.

We have now assembled the some of the definitions for Grothendieck's:

Theorem (Existence of the Hilbert Scheme): Fix a projective S-scheme X and a polynomial $P(n)$. Then the "Hilbert" functor $F_{X,P(n)}$ defined by:

$$F_{X,P(n)}(T) = \{\text{closed subschemes } Z \subset X \times T \text{ that are flat over } T \text{ with Hilbert polynomial } P(n)\},$$

$$F_{X,P(n)}(f) = \text{ "base extension by } f\text{"}$$

is represented by a **projective** S-scheme, denoted $H_{X,P(n)}$.

Obviously, we have some more background to fill in.

Reading: Hartshorne II.1-II.3.