3. Castelnuovo-Mumford Regularity and Spectral Sequences. This section is devoted to the following refinement of Serre's theorems A and B:

Uniform Vanishing Theorem: Fix the following data:

an infinite field k and a projective k-scheme X,

a locally free coherent sheaf $\mathcal{E}$ on X and a polynomial $P(d)$.

Then there is a d_0 depending only on $X, \mathcal{E}$ and $P(d)$ such that for all coherent subsheaves $\mathcal{F} \hookrightarrow \mathcal{E}$ with Hilbert polynomial $\chi(X, \mathcal{F}(d)) = P(d)$ and for all $d \geq d_0$, $\mathcal{F}(d)$ is generated by its sections and $\mathrm{H}^i(X, \mathcal{F}(d)) = 0$ for all $i > 0$.

As we will see, this is critical to the construction of the Hilbert scheme. Mumford's proof of this theorem uses the notion of:

Castelnuovo Regularity: A coherent sheaf $\mathcal{F}$ on $\mathbf{P}^n_k$ is m-regular if:

$$\mathrm{H}^i(\mathbf{P}^n_k, \mathcal{F}(m-i)) = 0 \text{ for all } i > 0$$

Examples: (a) $\mathcal{O}_{\mathbf{P}^n_k}$ is m-regular for $m \geq 0$, but not -1-regular.

(b) Suppose L is an invertible sheaf on a curve C such that:

(i) L is generated by it global sections,

(ii) $\mathrm{H}^1(C, L) = 0$, and

(iii) the multiplication map $\mathrm{Sym}^2\mathrm{H}^0(C, L) \to \mathrm{H}^0(C, L^2)$ is surjective.

Let $f : C \hookrightarrow \mathbf{P}^n_k$ be the closed immersion given by the complete linear series $\mathrm{H}^0(C, L)$. Then it follows from the exact sequences:

$$0 \to \mathcal{I}_C(d) \to \mathcal{O}_{\mathbf{P}^n_k}(d) \to f_*L^{\otimes d} \to 0$$

and (ii)-(iii) that $\mathcal{I}_C$ is 3-regular.

The Regularity Theorem: If $\mathcal{F}$ is m-regular, then:

(a) $\mathcal{F}$ is $m+1$-regular, and

(b) the multiplication map

$$H^0(\mathbf{P}^n_k, \mathcal{O}(1)) \otimes H^0(\mathbf{P}^n_k, \mathcal{F}(m)) \to H^0(\mathbf{P}^n_k, \mathcal{F}(m+1))$$

is surjective.

Corollary: If $\mathcal{F}$ is an m-regular coherent sheaf on $\mathbf{P}^n_k$, then:

(a) $\mathrm{H}^i(\mathbf{P}^n_k, \mathcal{F}(d)) = 0$ for all $i > 0$ and $d \geq m-1$ and

(b) $\mathcal{F}(d)$ is generated by global sections for all $d \geq m$.

Proof: Induction gives (a) immediately, and implies that the maps:

$$\mathrm{Sym}^e \mathrm{H}^0(\mathbf{P}^n_k, \mathcal{O}_{\mathbf{P}^n_k}(1)) \otimes \ \mathrm{H}^0(\mathbf{P}^n_k, \mathcal{F}(d)) \to \mathrm{H}^0(\mathbf{P}^n_k, \mathcal{F}(d+e))$$

are surjective for all $d \geq m$ and all $e > 0$. In particular, fix d and let e be large. By Serre's theorem B, the sheaf $\mathcal{F}(d+e)$ is generated by its sections, and it follows that $\mathcal{F}(d)$ is also generated by global sections.

Example: If L is a line bundle on a projective curve C satisfying (i)-(iii) of the example above, then $\mathrm{H}^1(C, L^d) = 0$ for all $d \geq 1$ and all multiplication maps $\mathrm{Sym}^d \mathrm{H}^0(C, L) \to \ \mathrm{H}^0(C, L^d)$ are surjective, i.e. the embedded curve is projectively normal. The corollary also tells us that $\mathcal{I}_C(3)$ is generated by its sections, hence C is an intersection of cubic hypersurfaces.

Mumford has a quick proof of the regularity theorem, but the funny-looking definition of regularity and the role of $\mathbf{P}^n_k$ seem to be better explained using spectral sequences, so we will prove it this way (following Mark Green).

Spectral Sequences: A **double complex** of abelian groups (or sheaves of abelian groups) is a collection $K_{p,q}$ of such groups (we'll assume $p, q \geq 0$), together with maps

$$d' : K_{p,q} \to K_{p+1,q} \text{ and } d'' : K_{p,q} \to K_{p,q+1}$$

satisfying $d' \circ d' = 0, d'' \circ d'' = 0$ and $d' \circ d'' + d'' \circ d' = 0$.

Given a double complex, one defines the associated **total complex**:

$$K_n := \bigoplus_{p+q=n} K_{p,q}, \ \ D = d' + d'' : K_n \to K_{n+1}$$

The basic homological algebra result asserts that the homology of the total complex:

$$E_n := \frac{\ker(D : K_n \to K_{n+1})}{\mathrm{im}(D : K_{n-1} \to K_n)}$$

filters in two different ways.

Starting from d': Let $E'^1_{p,q}$ be the homology of $K_{p,q}$ with respect to d'. Then d'' induces maps $d'_1 : E'^1_{p,q} \to E'^1_{p,q+1}$ satisfying $d'_1 \circ d'_1 = 0$. Inductively, if $E'^{r+1}_{p,q}$ is the homology of $E'^r_{p,q}$ with respect to $d'_r : E'^r_{p,q} \to E'^r_{p-r+1,q+r}$, then there are induced maps $d'_{r+1} : E'^{r+1}_{p,q} \to E'^{r+1}_{p-r,q+r+1}$.

Once $E'^r_{p,q} = E'^{r+1}_{p,q} = ...$, then we define $E'^\infty_{p,q} := E'^r_{p,q}$.

Starting from d'': The analog for d'' produces $d''_r : E''^r_{p,q} \to E''^r_{p+r,q-r+1}$ and we define $E''^\infty_{p,q}$ similarly.

The Spectral Sequence Theorem: There are two filtrations:

$$0 \subset F'_{n,n} \subset ... \subset F'_{n,q} \subset ... \subset F'_{n,0} = E_n \text{ and } 0 \subset ... \subset F''_{p,n} \subset ... \subset E_n$$

with subquotients $F'_{n,q}/F'_{n,q+1} \cong E'^\infty_{n-q,q}$ and $F''_{p,n}/F''_{p+1,n} \cong E''^\infty_{p,n-p}$.

Corollary: If $E'^\infty_{p,q} = 0$ for all p, q, then $E''^\infty_{p,q} = 0$ for all p, q.

Now suppose that $0 \to \mathcal{F}_0 \to \mathcal{F}_1 \to ... \to \mathcal{F}_m \to 0$ is a long **exact** sequence of coherent sheaves on a Noetherian separated scheme X. Take an open cover $\mathcal{U}$ of X by affines $U_0, ..., U_n$ and consider the double complex:

$$K_{p,q} = C^q(\mathcal{U}, \mathcal{F}_p) := \bigoplus_{i_0 < ... < i_q} \mathcal{F}_p(U_{i_0} \cap ... \cap U_{i_q})$$

with $(-1)^p d''$ the Cech coboundary map and d' induced from the sheaf maps.

Because the sequence of sheaves is exact, it follows that $E'^\infty_{p,q} = E'^1_{p,q} = 0$ for all p, q. So by the theorem, it must be true that $E''^\infty_{p,q} = 0$ as well. Since the terms $E''^1_{p,q} = \mathrm{H}^q(X, \mathcal{F}_p)$ are in general not zero, this gives us information about maps between cohomology groups of the $\mathcal{F}_p$.

Exercise: Apply this to a short exact sequence of sheaves to obtain the connecting homomorphisms from the maps d''_2.

The **Koszul complex** $\mathcal{K}_*$ is the exact sequence:

$$0 \to \wedge^{n+1} V \otimes_k \mathcal{O}(-n-1) \to \wedge^n V \otimes_k \mathcal{O}(-n) \to ... \to V \otimes_k \mathcal{O}(-1) \to \mathcal{O} \to 0$$

of locally free sheaves on $\mathbf{P}^n_k$, where V is the vector space $\mathrm{H}^0(\mathbf{P}^n_k, \mathcal{O}_{\mathbf{P}^n_k}(1))$ with basis $x_0, ..., x_n$ and $\wedge^r V \otimes_k \mathcal{O}(-r) \to \wedge^{r-1} V \otimes_k \mathcal{O}(-r+1)$ are the contraction maps:

$$x_{i_1} \wedge ... \wedge x_{i_r} \otimes s \to \sum (-1)^{j-1} x_{i_1} \wedge ... \wedge \widehat{x_{i_j}} \wedge ... \wedge x_{i_r} \otimes x_{i_j} s$$

Proof of the Regularity Theorem: Let $\mathcal{K}_* \otimes_{\mathcal{O}_{\mathbf{P}^n_k}} \mathcal{F}(m+1)$ be the complex of sheaves obtained by tensoring the terms of the Koszul complex by $\mathcal{F}(m+1)$. Because the terms in the Koszul complex are locally free, it follows that this complex is exact. One now computes: $E''^1_{n,0} = V \otimes_k \mathrm{H}^0(\mathbf{P}^n_k, \mathcal{F}(m))$, $E''^1_{n+1,0} = \mathrm{H}^0(\mathbf{P}^n, \mathcal{F}(m+1))$ and d''_1 is the multiplication map. Furthermore, each $E''^1_{n-r,r} = \wedge^{r+1} V \otimes \mathrm{H}^r(\mathbf{P}^n, \mathcal{F}(m-r))$ that might eventually produce an $E''^r_{n-r,r}$ mapping to $E''^r_{n+1,0}$ under d''_r is already zero, since we assumed $\mathcal{F}$ was m-regular. Since all d''_r maps out of $E''^r_{n+1,0}$ are obviously zero as well, the vanishing of $E''^\infty_{n+1,0}$ forces the surjectivity of d''_1, giving us (b).

Similarly, the vanishings $\mathrm{H}^i(\mathbf{P}^n_k, \mathcal{O}_{\mathbf{P}^n_k}(m+1-i))$ are seen by considering the long exact sequences $\mathcal{K}_* \otimes_{\mathcal{O}_{\mathbf{P}^n_k}} \mathcal{F}(m+1-i)$ for each $1 \leq i \leq n$.

Proof of the Uniform Vanishing Theorem: We prove the theorem first in case $X = \mathbf{P}^n_k$, $\mathcal{F}$ is locally free, and $\mathcal{E} = \oplus \mathcal{O}_{\mathbf{P}^n_k}(l)$ for some l. We'll work by induction on n, with the case $n = 0$ being trivial.

An injective map $i : \mathcal{F} \hookrightarrow \mathcal{E}$ of locally free sheaves need not be injective on all fibers, but it does need to be injective on fibers at all generic points, hence at all points $x \in U$ of a dense open set, by Nakayama's lemma. The converse also holds. A map of locally free sheaves on an integral scheme is injective if and only if it is injective on fibers at some point.

Now suppose $f : H \hookrightarrow \mathbf{P}^n_k$ is the hyperplane corresponding to the section $s \in \mathrm{H}^0(\mathbf{P}^n_k, \mathcal{O}_{\mathbf{P}^n_k}(1))$. Then the following sequences are exact for all d:

$$(*)\ 0 \to \mathcal{F}(d-1) \xrightarrow{s} \mathcal{F}(d) \to f_*\mathcal{F}(d)|_H \to 0$$

where $\mathcal{F}|_H := f^*\mathcal{F}$. Moreover, if H passes through a point where i is injective on fibers, then $i|_H : \mathcal{F}|_H \to \mathcal{E}|_H \cong \oplus \mathcal{O}_H(l)$ is injective, as well. We will argue by induction using such an H.

Suppose $P(d) = \chi(\mathbf{P}^n, \mathcal{F}(d)) = \sum_{i=0}^n a_i \binom{d}{i}$. Then

$$\begin{aligned} \chi(H, \mathcal{F}|_H(d)) &= \textstyle\sum_{i=0}^n a_i \binom{d}{i} - a_i \binom{d-1}{i} \\ &= \textstyle\sum_{i=0}^{n-1} a_{i+1} \binom{d-1}{i} \end{aligned}$$

Thus the restrictions $\mathcal{F}|_H$ all have the same Hilbert polynomial and all are subsheaves of the fixed locally free sheaf $\mathcal{E}|_H = \oplus \mathcal{O}_H(l)$.

So we may assume that there is an N_0 independent of $\mathcal{F}$ such that

$$\mathrm{H}^i(H, \mathcal{F}_H(d)) = 0 \text{ for all } i > 0, d \geq N_0$$

From the exact sequence on cohomology associated to $(*)$, this gives us:

$$\mathrm{H}^i(\mathbf{P}^n_k, \mathcal{F}(N_0 - 1)) = \mathrm{H}^i(\mathbf{P}^n_k, \mathcal{F}(N_0)) = \ldots \text{ forall } i \geq 2$$

and so by Serre's theorem B, we get $\mathrm{H}^i(\mathbf{P}^n_k, \mathcal{F}(d)) = 0$ for $i \geq 2, d \geq N_0 - 1$. To get the H^1 term, we use the cohomology sequences for $d \geq N_0$:

$$0 \to \mathrm{H}^0(\mathcal{F}(d-1)) \to \mathrm{H}^0(\mathcal{F}(d)) \xrightarrow{\rho_d} \mathrm{H}^0(\mathcal{F}_H(d)) \to$$

$$\to \mathrm{H}^1(\mathcal{F}(d-1)) \to \mathrm{H}^1(\mathcal{F}(d)) \to 0$$

Since all the higher cohomology of $\mathcal{F}_H(N_0)$ and higher twists vanishes, it follows in particular that $\mathcal{F}_H$ is $N_1 := N_0 + n - 1$ regular, so by the regularity theorem, the multiplication map:

$$\mathrm{H}^0(\mathcal{O}_H(1)) \otimes \mathrm{H}^0(\mathcal{F}_H(d)) \to \mathrm{H}^0(\mathcal{F}_H(d+1))$$

is surjective for $d \geq N_1$. Thus if the restriction map ρ_d is surjective for some $d \geq N_1$, then the restriction map ρ_{d+1} must also be surjective, as is evident from the following diagram:

$$\begin{array}{ccc} \mathrm{H}^0(\mathcal{O}_{\mathbf{P}^n_k}(1)) \otimes \mathrm{H}^0(\mathcal{F}(d)) & \to & \mathrm{H}^0(\mathcal{F}(d+1)) \\ \downarrow & & \downarrow \\ \mathrm{H}^0(\mathcal{O}_H(1)) \otimes \mathrm{H}^0(\mathcal{F}_H(d)) & \to & \mathrm{H}^0(\mathcal{F}_H(d+1)) \end{array}$$

On the other hand, $\dim_k \mathrm{H}^1(\mathcal{F}(d-1)) > \dim_k \mathrm{H}^1(\mathcal{F}(d))$ whenever ρ_d is not surjective, so the dimensions of $\mathrm{H}^1(\mathcal{F}(d))$ are strictly decreasing down to zero for $d \geq N_1 - 1$. In particular, **all** the higher cohomology of $\mathcal{F}(d)$ vanishes as soon as $d \geq N_2 := N_1 + \dim_k \mathrm{H}^1(\mathcal{F}(N_1 - 1))$. We now need to show that this can be bounded independent of $\mathcal{F}$.

Since $\mathrm{H}^i(\mathcal{F}(N_1 - 1)) = 0$ for $i \geq 2$, we have:

$$\begin{aligned} \dim_k \mathrm{H}^1(\mathcal{F}(N_1 - 1)) &= \dim_k \mathrm{H}^0(\mathcal{F}(N_1 - 1)) - \chi(\mathcal{F}(N_1 - 1)) \\ &\leq \dim_k \mathrm{H}^0(\mathcal{E}(N_1 - 1)) - P(N_1 - 1) \end{aligned}$$

which indeed only depends upon $P(d)$ and $\mathcal{E}$.

Not only do we have vanishing of higher cohomology when $d \geq N_2$, but we also see that $\mathcal{F}$ is $N_2 + 1$ regular. Hence the $\mathcal{F}(d)$ are generated by their sections for all $d \geq d_0 := N_2 + 1$ and we get uniform vanishing.

Next, suppose $\mathcal{F}$ is not assumed to be locally free. Then the proof above holds provided there is always a hyperplane $H \subset \mathbf{P}_k^n$ so that the sequences:

$$(**)\ 0 \to \mathcal{F}(d-1) \to \mathcal{F}(d) \to \mathcal{F}_H(d) \to 0$$

are exact, and $i_H : \mathcal{F}_H \to E_H$ is injective.

For this, we need a result from commutative algebra. If $\mathcal{F}$ is a sheaf and $s \in \mathcal{F}(U)$ is a section over an open set U, then the **support** of s in U is the set of points $x \in U$ such that $0 \neq s_x \in \mathcal{F}_x$. Note that this is always closed. If $\mathcal{F}$ is a coherent sheaf over a Noetherian scheme, then we say $x \in X$ is an **associated point** of $\mathcal{F}$ if there exist an s and $x \in U$ as above so that x is in the support of s and moreover, the support of s is the closure(!) of x in U.

For example, the generic point(s) are the only possible associated points of a subsheaf of a locally free sheaf on a reduced scheme. The extra associated points of $\mathcal{O}_X$ on a non-reduced scheme X are called the **embedded primes**.

If $X = \operatorname{Spec}(A)$ is affine and $\mathcal{F} = \widetilde{M}$, then it follows from the primary decomposition of the sub-module $0 \subset M$ (see Atiyah-MacDonald Ch. 4,7) that there are only finitely many associated points of $\mathcal{F}$, and that if $a \in A$ does not belong to the union of the corresponding prime ideals in A, then $am \neq 0$ for all $m \in M$. What we glean from this in general is:

$$\mathrm{A}(\mathcal{F}) := \{\text{associated points of } \mathcal{F}\} \text{ is a finite set}$$

Returning to the proof, notice that since $\mathcal{F}$ is a subsheaf of a locally free sheaf, $\mathcal{F}$ has only the generic point for an associated point, hence $(**)$ is exact for any choice of H. If we choose H to be disjoint from the finite set $\mathrm{A}(\mathcal{E}/\mathcal{F})$, then the equation for H is not a zero divisor in any stalk of $\mathcal{E}/\mathcal{F}$. Let $\mathcal{O}_x$ be the stalk of $\mathcal{O}_{\mathbf{P}_k^n}$ at x and s_x be the equation for H at x. Then on stalks at x the map i_H is the first map of the sequence:

$$\mathcal{F}_x \otimes_{\mathcal{O}_x} \mathcal{O}_x/s_x \to \mathcal{E}_x \otimes_{\mathcal{O}_x} \mathcal{O}_x/s_x \to \mathcal{E}_x/\mathcal{F}_x \otimes_{\mathcal{O}_x} \mathcal{O}_x/s_x \to 0$$

which has kernel $\mathrm{Tor}^1_{\mathcal{O}_x}(\mathcal{E}_x/\mathcal{F}_x, \mathcal{O}_x/s_x)$. But since our choice of H guarantees that multiplication of $\mathcal{E}_x/\mathcal{F}_x$ by s_x has no kernel, it follows that this Tor is zero, and that i_H is injective.

Finally, to prove the theorem on any projective k-scheme $f : X \hookrightarrow \mathbf{P}^n_k$ and for any locally free $\mathcal{E}$ on X, we first note that by Serre's theorem A, we can always inject $\mathcal{E} \hookrightarrow \oplus\mathcal{O}_X(l)$ for some l. Thus, we may as well assume that $\mathcal{E} = \oplus\mathcal{O}_X(l)$. Then for any $\mathcal{F} \hookrightarrow \oplus\mathcal{O}_X(l)$, there is a diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathcal{K}_1 & = & \mathcal{K}_1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & \mathcal{K}_2 & \to & \oplus\mathcal{O}_{\mathbf{P}^n_k}(l) & \to & f_*(\oplus\mathcal{O}_X(l)/\mathcal{F}) & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & f_*\mathcal{F} & \to & \oplus\mathcal{O}_X(l) & \to & f_*(\oplus\mathcal{O}_X(l)/\mathcal{F}) & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Note that $\mathcal{K}_1$ and $\mathcal{K}_2$ are both subsheaves of $\oplus\mathcal{O}_{\mathbf{P}^n_k}(l)$ with Hilbert polynomials only depending upon $X, \mathcal{E}$ and $P(d)$. Thus uniform vanishing applies to both $\mathcal{K}_1$ and $\mathcal{K}_2$. From the exact sequences:

$$\to \mathrm{H}^i(\mathbf{P}^n_k, \mathcal{K}_2(d)) \to \mathrm{H}^i(X, \mathcal{F}(d)) \to \mathrm{H}^{i+1}(\mathbf{P}^n_k, \mathcal{K}_1(d)) \to$$

and the surjectivity:

$$\mathcal{K}_2(d) \to f_*(\mathcal{F}(d))$$

it follows that uniform vanishing holds for $\mathcal{F}$ with d_0 equal to the maximum of the values for $\mathcal{K}_1$ and $\mathcal{K}_2$. This completes the proof.

Remark: The only place we used the fact that $\mathcal{E}$ was locally free was in the assertion that there is an injection:

$$\mathcal{E} \hookrightarrow \oplus\mathcal{O}_X(l)$$

This injection is achieved by applying Serre's theorem A to the dual locally free sheaf $\mathcal{E}^* = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X)$ to find a surjection $\oplus\mathcal{O}_{\mathbf{P}^n_k} \to \mathcal{E}^*(l)$, twisting by $\mathcal{O}_{\mathbf{P}^n_k}(-l)$ and then dualizing back. Thus it certainly suffices to assume that $\mathcal{E}^{**} \cong \mathcal{E}$ (i.e. $\mathcal{E}$ is **reflexive**), or even just that $\mathcal{E}$ injects into $\mathcal{E}^{**}$ (i.e. $\mathcal{E}$ is **torsion-free**), since $\mathcal{E}^{**}$ is always reflexive.