4. Flatness. Our first goal is to prove and generalize the following:

Properness Theorem: Let S be a nonsingular curve, and let $U \subset S$ be the punctured curve $S - s_0$. Then every family of projective schemes $X_U \subset \mathbf{P}^n_U$ with constant Hilbert polynomial $P(d)$ extends uniquely to a family $X_S \subset \mathbf{P}^n_S$ such that the Hilbert polynomial of the "limit" $X_{s_0} \subset \mathbf{P}^n_{k(s_0)}$ is also $P(d)$.

The fibers of an S-scheme $f : X \to S$ are $X_s := X \times_S \operatorname{Spec}(k(s)) \subset X$, where $\operatorname{Spec}(k(s)) \to S$ "is" the inclusion of the point $s \in S$. These fibers are the associated **family** of subschemes of X parametrized by S. Similarly, a sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules determines the family $\mathcal{F}_s := \mathcal{F}_s|_{X_s}$ of sheaves of $\mathcal{O}_{X_s}$-modules parametrized by S. Thus $\mathbf{P}^n_S$ is the family of projective spaces $\mathbf{P}^n_{k(s)}$ and if a closed immersion $X \subset \mathbf{P}^n_S$ is given, then the subschemes $X_s \subset \mathbf{P}^n_{k(s)}$ and sheaves $\mathcal{F}_s$ have Hilbert polynomials depending upon the point $s \in S$.

Grothendieck first recognized the usefulness of the algebraic notion of flatness in defining a "good" family of schemes, or of coherent sheaves.

$\mathcal{F}$ is **flat over** S **at** $x \in X$ if the stalk $\mathcal{F}_x$ is a flat $\mathcal{O}_{f(x)}$-module. If $\mathcal{F}$ is flat over S at every point of X, we say that $\mathcal{F}$ is **flat over** S. We say that the morphism f is flat if $\mathcal{O}_X$ is flat over S.

"Classical" smooth families are flat. That is, if $f : X \to Y$ is a morphism of nonsingular varieties over an algebraically closed field k, with the property that $\dim(X) - \dim(Y) = n$ and $\Omega_{X/Y}$ is locally free of rank n, then f is flat.

If Y is a Noetherian scheme and $f : X \to Y$ is a finite map, then a coherent sheaf $\mathcal{F}$ is flat over Y if and only if $f_*\mathcal{F}$ is locally free. This is a consequence of the fact that a finitely generated module over a Noetherian local ring is flat if and only if it is free.

For the properness theorem, the key property of flatness is the following:

Constancy of the Hilbert Polynomial: If A is a Noetherian integral domain, X is a projective A-scheme with very ample invertible sheaf $\mathcal{O}_X(1)$, and $\mathcal{F}$ is a coherent sheaf on X, then the following are equivalent:

(a) $\mathcal{F}$ is flat over $\operatorname{Spec}(A)$

(b) $\mathrm{H}^0(X, \mathcal{F}(d))$ is a locally free A-module, for all $d >> 0$

(c) The Hilbert polynomial $\chi(X_s, \mathcal{F}_s(d))$ is independent of $s \in \operatorname{Spec}(A)$.

Proof: (a) $\Rightarrow$ (b). Let $\mathcal{U}$ be an affine cover of X. By Serre's theorem B, the higher cohomology of $\mathcal{F}(d)$ vanishes for $d >> 0$, hence the following Cech complexes are **exact** for large d:

$$0 \to \mathrm{H}^0(X, \mathcal{F}(d)) \to C^0(\mathcal{U}, \mathcal{F}(d)) \to C^1(\mathcal{U}, \mathcal{F}(d)) \to ... \to C^n(\mathcal{U}, \mathcal{F}(d)) \to 0$$

But flatness of $\mathcal{F}$ implies that each term in the Cech complex is a flat A-module, hence $\mathrm{H}^0(X, \mathcal{F}(d))$ is also flat, and since it is finitely generated, it is locally free.

(b) $\Rightarrow$ (a). Let $f : X \to \mathbf{P}^n_A$ be the closed immersion. Then evidently $\mathcal{F}$ is flat over $\mathrm{Spec}(A)$ if and only if $f_*\mathcal{F}$ is flat over $\mathrm{Spec}(A)$. Let

$$M = \bigoplus_{d \geq d_0} \Gamma(\mathbf{P}^n_A, f_*\mathcal{F}(d))$$

where d_0 is chosen so that $\mathrm{H}^0(X, \mathcal{F}(d))$ is locally free for all $d \geq d_0$. Then M is flat, and $\widehat{M} = f_*\mathcal{F}$, hence $f_*\mathcal{F}$ is flat.

(b) $\Rightarrow$ (c). We will prove that for each $s \in S$ and all $d >> 0$, there are isomorphisms:

$$(*) \quad \mathrm{H}^0(X, \mathcal{F}(d)) \otimes_A k(s) \xrightarrow{\sim} \mathrm{H}^0(X_s, \mathcal{F}_s(d))$$

Then (b) implies that $\chi(X_s, \mathcal{F}_s(d))$ is the rank of the locally free A-module $\mathrm{H}^0(X, \mathcal{F}(d))$ at s for large d, hence independent of s.

If $s \in \mathrm{Spec}(A)$ is a closed point, then given a presentation of $k(s) \cong A/m$:

$$A^n \to A \to k(s) \to 0$$

we get a presentation of $\mathrm{H}^0(X, \mathcal{F}(d)) \otimes_A k(s)$ by tensoring with $\mathrm{H}^0(X, \mathcal{F}(d))$, or we can sheafify the given presentation and tensor with $\mathcal{F}$ over $\mathcal{O}_X$ to get a presentation of $\mathcal{F}_s(d)$. For $d >> 0$, this yields a presentation of $\mathrm{H}^0(X_s, \mathcal{F}_s(d))$ identical to the other.

In general, one uses a "flat base extension" to reduce to the case where s is a closed point. That is, construct the fiber product:

$$\begin{array}{ccc} X' & \to & X \\ \downarrow & & \downarrow \\ \mathrm{Spec}(A_{\mathcal{P}_s}) & \to & \mathrm{Spec}(A) \end{array}$$

where $\mathcal{P}_s$ is the prime in A corresponding to s. Since localizing is exact, and each Cech complex on X localizes to a Cech complex on X', it follows that:

$$\mathrm{H}^i(X, \mathcal{F}) \otimes_A A_{\mathcal{P}_s} \xrightarrow{\sim} \mathrm{H}^i(X', \mathcal{F}')$$

for all i and all coherent sheaves $\mathcal{F}$ on X ($\mathcal{F}'$ is the pull-back of $\mathcal{F}$ to X'). But now the fiber of X' over the closed point of $\mathrm{Spec}(A_{\mathcal{P}_s})$ is X_s, and we have reduced to the previous case.

(c) $\Rightarrow$ (a). Suppose $\chi(X_s, \mathcal{F}_s(d)) = \chi(X_\xi, \mathcal{F}_\xi(d))$ where $\xi \in S$ is the generic point. Then by $(*)$ and Serre's theorem B, we know that $\mathrm{H}^0(X, \mathcal{F}(d))$ has the same rank at s as it has at the generic point, for all large enough d. By Nakayama's lemma, this implies that for such large d, $\mathrm{H}^0(X, \mathcal{F}(d)) \otimes_A A_{\mathcal{P}_s}$ is a free $A_{\mathcal{P}_s}$-module. As in the proof of (b) $\Rightarrow$ (a), this implies that $\mathcal{F}$ is flat over S at all points mapping to s.

Corollary: Suppose S is a Noetherian connected scheme and $X \subset \mathbf{P}^n_S$ is a projective S-scheme which is flat over S. Then:

(a) The fiber dimensions $\dim(X_s)$ are constant

(b) The Euler characteristic $\chi(X_s, \mathcal{O}_{X_s})$ is constant, and

(c) The degree of $X_s \subset \mathbf{P}^n_s$ is constant.

Proof: (a)-(c) are obtained from the Hilbert polynomial as the degree, the constant term and the leading coefficient, respectively. Any two points on a connected scheme can be joined by a sequence of integral subschemes.

Remark: There are at least two good reasons to use flatness instead of the equivalent constancy of the Hilbert polynomial. First, flatness gives the "correct" generalization to projective schemes over schemes that are not reduced (this will be particularly important in deformation theory) and secondly, flatness is an easier notion to work with. As an example, we will use it to prove the properness theorem by making use of the particularly simple description of flatness over regular rings of dimension one.

Flatness over Nonsingular Curves 1: If $f : X \to S$ and S is irreducible and regular of dimension one, then a sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules is flat over S if and only if f maps each associated point of $\mathcal{F}$ to the generic point of S.

E.g. If X is reduced, then $f : X \to S$ is a flat morphism if and only if every irreducible component of X dominates S.

Proof: If f is flat and $x \in X$ is a point with closed image $s = f(x) \in S$, then $\mathcal{O}_{S,s}$ is a DVR. Let $t \in m_s$ be a uniformizing parameter, which is in particular not a zero divisor in $\mathcal{O}_{S,s}$. Then $f^{\#}t \in m_x$ cannot be a zero divisor in the flat module $\mathcal{F}_x$. Hence x is not an associated point of $\mathcal{F}$.

Conversely, any $\mathcal{F}$ is flat at all points mapping to the generic point of S because the local ring at the generic point is a field(!) If $f(x) = s$ is closed and $\overline{\{x\}}$ does not contain an associated point of $\mathcal{F}$, then multiplication by the uniformizing parameter $f^{\#}t$ is injective on $\mathcal{F}_x$, and since t generates m_s, it follows that $\mathcal{F}_x$ is flat over $\mathcal{O}_{S,s}$.

Remark: The previous result is certainly false if S is singular. For example, the normalization map $f : X \to S$ of a nodal curve is obviously not flat.

Flatness over Nonsingular Curves 2: If S is irreducible and regular of dimension one and $U \subset S$ is the complement of a closed point $s_0 \in S$, then "flat quotients uniquely extend across s_0". That is, given $f : X \to S$ and a coherent sheaf $\mathcal{E}$ on X, let $X_U = f^{-1}(U)$ and $\mathcal{E}_U = \mathcal{E}|_{X_U}$. Then a surjection $\mathcal{E}_U \to \mathcal{Q}$ to a coherent sheaf $\mathcal{Q}$ that is flat over U extends uniquely to a surjection $\mathcal{E} \to \overline{\mathcal{Q}}$ where $\overline{\mathcal{Q}}$ is coherent and flat over S.

Proof: Indeed, we will prove that $\mathcal{E} \to \overline{\mathcal{Q}}$ is universal in the sense that if $\mathcal{E} \to \mathcal{Q}'$ is another surjection extending $\mathcal{E}|_U \to \mathcal{Q}$, then there is a (unique) factorization $\mathcal{E} \to \mathcal{Q}' \to \overline{\mathcal{Q}}$

Note that if such a universal extension exists, then $\overline{\mathcal{Q}}$ is flat. Otherwise there would be a section of $\overline{\mathcal{Q}}$ with support lying over s_0 by Flatness over Curves 1, and the further quotient of $\overline{\mathcal{Q}}$ by the subsheaf generated by this section would violate universality. Similarly, given another $\mathcal{Q}'$, the kernel of the induced map $\mathcal{Q}' \to \overline{\mathcal{Q}}$ would be non-trivial, giving associated points in the fiber of f over s_0. Thus using Flatness over Curves 1 again, we see that the universal extension *if it exists* is the unique flat extension.

So we have to prove that such a universal extension exists. It obviously suffices to do this in case $S = \mathrm{Spec}(B)$ is affine and $U = \mathrm{Spec}(B_g)$ for some $g \in B$, but it also suffices to do this in case $X = \mathrm{Spec}(A)$, since the extensions over affines will patch together by the universal property. In this case, $X_U = \mathrm{Spec}(A_g)$, $\mathcal{E} = \widetilde{M}$, $\mathcal{E}_U = \widetilde{M_g}$, $\mathcal{Q}_U = \widetilde{N}$, and we are given $M_g \to N$, a surjective map of A_g-modules. We should take $\overline{N}$ to be the image of the A-module map $M \to M_g \to N$ and $\mathcal{Q}_S = \widetilde{\overline{N}}$. One then readily checks that $\widetilde{M} \to \widetilde{\overline{N}}$ is the universal extension of $\widetilde{M_g} \to \widetilde{N}$.

Example: In the case $\mathcal{E} = \mathcal{O}_X$, then all surjective maps are of the form $\mathcal{O}_X \to i_*\mathcal{O}_Z$, where $i : Z \hookrightarrow X$ is a closed immersion of S-schemes. The proof of Flatness over Curves 2 in this context shows that every closed subscheme $Y \subset X_U$ has a unique smallest "scheme-theoretic closure" $\overline{Y} \subset X$ and that if Y is flat over U, then $\overline{Y}$ is flat over S.

Putting the results of this section together, we now have a:

Proof of the Properness Theorem: If $\mathcal{E}$ is a coherent sheaf on $\mathbf{P}^n_S$ and a surjection $\mathcal{E}|_U \to \mathcal{Q}$ is given, such that the Hilbert polynomial of $\mathcal{Q}_s$ is constant for $s \in U$, then by the Constancy of the Hilbert Polynomial, $\mathcal{Q}$ is flat over U. By Flatness over Curves 2, there is a unique extension $\mathcal{E} \to \overline{\mathcal{Q}}$ such that $\overline{\mathcal{Q}}$ is flat over S, and by the Constancy result again, $\mathcal{Q}_{s_0}$ has the same Hilbert polynomial. When we apply this to the case $\mathcal{E} = \mathcal{O}_{\mathbf{P}^n_S}$ and $\mathcal{Q} = i_*\mathcal{O}_{X_U}$ for the closed immersion $i : X_U \hookrightarrow \mathbf{P}^n_U$, we get the theorem.

The next result is a sort of Sard's theorem for flatness:

Generic Flatness: Given a reduced Noetherian scheme S, a morphism $f : X \to S$ of finite type, and a coherent sheaf $\mathcal{F}$ on X, then there is a non-empty open subset $U \subset S$ such that $\mathcal{F}_U$ is flat over U, where:

$$X_U := f^{-1}(U) = X \times_S U \quad \text{and} \quad \mathcal{F}_U := \mathcal{F}|_{X_U}$$

Proof: We may assume $S =$ Spec(A) where A is an integral domain, replacing S by an open set. We may also assume f is dominant, otherwise generic flatness would be trivially true with $U = S - \overline{f(X)}$. More generally, we can always ignore locally closed subsets of X that do not dominate S by shrinking S if necessary.

Cover X by finitely many open affines Spec(B) and write $\mathcal{F}|_{\text{Spec}(B)} = \widetilde{M}$. Then it suffices to prove generic flatness for each Spec(B) that dominates Spec(A). From the existence of a filtration of M:

$$0 \subset M_1 \subset M_2 \subset .... \subset M_n = M \text{ with } M_{i+1}/M_i \cong B/\mathcal{P}_i$$

it suffices to prove generic flatness for $M = B/\mathcal{P}$, and moreover for those $B/\mathcal{P}$ with the property that the map $A \to B/\mathcal{P}$ is injective, again shrinking S if necessary. Thus it suffices to prove the following:

Claim: If A is a Noetherian integral domain and B is an integral domain which is finitely generated as an A-algebra, then there is an $f \in A$ with the property that B_f is a free A_f-algebra.

Proof: Let n be the transcendence degree of B over A and let $K = K(A)$ be the field of fractions of A. Then the ring $K \otimes_A B$ is finitely generated as a K-algebra, hence by Noether Normalization it is integral over a polynomial ring $K[f_1, ..., f_n]$ with $f_1, ..., f_n \in B$. It does not follow that B is integral over $A[f_1, ..., f_n]$, but if B is generated by $b_1, ..., b_r$ as an A-algebra, then there are only finitely many denominators (in A) appearing in the minimal polynomials of the $b_i \in K \otimes_A B$ over $K[f_1, ..., f_n]$. If f is taken to be the product of the denominators, then it follows that B_f is integral over the polynomial ring $A_f[f_1, ..., f_n]$, hence of finite rank as an $A_f[f_1, ..., f_n]$-module. If $m = \mathrm{rk}(B_f)$, then there is an exact sequence of $A_f[f_1, ..., f_n]$-modules:

$$0 \to A_f[f_1, ..., f_n]^m \to B_f \to Q \to 0$$

where Q is a torsion module. Taking a filtration of Q as we did earlier with M, we return to the claim with B' an integral domain over A of smaller transcendence degree. By induction, we may therefore assume that Q_g is a free A_g-module, and it follows that B_{fg} is a free A_{fg}-module.

Corollary: In the same setting, there is a stratification:

$$S = \coprod V_i$$

of S by reduced, locally closed subschemes V_i such that $\mathcal{F}_{V_i}$ is flat, for all i.

Proof: Noetherian induction.

Corollary: If in addition $f : X \to S$ is a projective morphism, with closed immersion $X \subset \mathbf{P}^n_S$, then the Hilbert polynomials of the $\mathcal{F}_s$ vary in finite set.

Proof: Apply the constancy result to the components of the V_i strata.

This stratification is not quite good enough for our purposes. We need:

Flattening Stratifications: Given a projective morphism $f : X \to S$ over a Noetherian scheme S and a coherent sheaf $\mathcal{F}$ on X, there is a (unique) stratification:

$$S = \coprod S_i$$

of S by locally closed subschemes S_i such that $\mathcal{F}_{S_i}$ is flat over each S_i and more generally, given a morphism $g : T \to S$ and hence $\widetilde{g} : X \times_S T \to X$, then $\widetilde{g}^*\mathcal{F}$ is flat over T **if and only if** g factors through:

$$g : T \to \coprod S_i \to S$$

(Roughly, the S_i are the maximal thickenings of the V_i preserving flatness)

Proof: We may as well assume that $S = \mathrm{Spec}(A)$ since flattening stratifications for an open cover will glue by their universal property. And it clearly suffices to consider only affine $T = \mathrm{Spec}(B)$ as well. Given a morphism $g : T \to S$, we will write the fiber square as:

$$\begin{array}{ccc} X_B & \stackrel{\widetilde{g}}{\to} & X_A \\ \widetilde{f} \downarrow & & f \downarrow \\ \mathrm{Spec}(B) & \stackrel{g}{\to} & \mathrm{Spec}(A) \end{array}$$

Any Cech complex of A-modules:

$$0 \to \mathrm{H}^0(X_A, \mathcal{E}_A) \to C^0(\mathcal{U}, \mathcal{E}_A) \to ... \to C^n(\mathcal{U}, \mathcal{E}_A) \to 0$$

for a quasi-coherent sheaf $\mathcal{E}_A$ on X_A becomes a Cech complex for $\mathcal{E}_B := \widetilde{g}^*\mathcal{E}$ when we tensor it by B. It therefore follows from the right-exactness of the tensor product that there are always maps:

$$\mathrm{H}^i(X_A, \mathcal{E}_A) \otimes_A B \to \mathrm{H}^i(X_B, \mathcal{E}_B)$$

for all i. Moreover, if B is flat over A, then all these maps are isomorphisms. This generalizes an earlier "flat base extension" argument.

We will need the following two "base change" theorems:

Stable Base Change: If g is fixed and $d >> 0$, then:

$$\mathrm{H}^0(X_A, \mathcal{F}_A(d)) \otimes_A B \to \mathrm{H}^0(X_B, \mathcal{F}_B(d)) \text{ is an isomorphism}$$

Cohomology and Base Change: If $f : Y \to \mathrm{Spec}(B)$ is a projective morphism and $\mathcal{E}$ is a coherent sheaf on Y which is flat over $\mathrm{Spec}(B)$, let $t \in \mathrm{Spec}(B)$ be a point. Then:

(a) If $\phi^i(t) : \mathrm{H}^i(Y, \mathcal{E}) \otimes_B k(t) \to \mathrm{H}^i(Y_t, \mathcal{E}_t)$ is surjective, then it is an isomorphism and $\phi^i(t')$ is an isomorphism for all t' in a neighborhood of t.

(b) If $\phi^i(t)$ is surjective, then the following are equivalent:

(I) $\phi^{i-1}(t)$ is also surjective.

(II) $\mathrm{H}^i(Y, \mathcal{E})$ is locally free in a neighborhood of t.

Corollary: If $\mathrm{H}^1(Y_t, \mathcal{E}_t) = 0$ for all fibers, then $\mathrm{H}^0(Y, \mathcal{E})$ is locally free.

Now use the weak stratification coming from generic flatness to pull back $\mathcal{F}$ to coherent sheaves $\mathcal{F}_{V_i}$ on each $X_{V_i} := X \times_S V_i$ that are flat over V_i. If $\mathrm{Spec}(A_i) \subset V_i$ is an affine open set, then as in the earlier proof of (a) $\Rightarrow$ (b), the Cech complex for large d:

$$0 \to \mathrm{H}^0(X_{A_i}, \mathcal{F}_{A_i}(d)) \to C^0(\mathcal{U}, \mathcal{F}_{A_i}(d)) \to ... \to C^n(\mathcal{U}, \mathcal{F}_{A_i}(d)) \to 0$$

is an exact sequence of flat A_i-modules. But because of flatness, if we tensor the sequence by $k(s)$ for $s \in \mathrm{Spec}(A_i)$, the sequence remains exact. Since it is then a Cech complex for $\mathcal{F}_s(d)$ on the fiber X_s, this shows that the higher cohomology of the $\mathcal{F}_s(d)$ for $s \in \mathrm{Spec}(A_i)$ all vanish for such d. Choosing d_0 large enough to work for all A_i simultaneously, we therefore obtain:

(i) $\mathrm{H}^i(X_s, \mathcal{F}_s(d)) = 0$ for all $s \in S$ and $d \geq d_0$.

Moreover, we can use the stable base change result to conclude that:

$$\mathrm{H}^0(X_A, \mathcal{F}_A(d)) \otimes_A A_i \tilde{\to} \mathrm{H}^0(X_{A_i}, \mathcal{F}_{A_i}(d) \text{ for all } A_i \text{ and } d \geq d_0$$

increasing d_0 if necessary. From the exact sequence above or the cohomology and base change result, we may then conclude:

(ii) $\mathrm{H}^0(X_A, \mathcal{F}_A(d)) \otimes_A k(s) \tilde{\to}\ \mathrm{H}^0(X_s, \mathcal{F}_s(d))$ for all $s \in S$ and $d \geq d_0$.

This is a uniform version of the result $(*)$ which we proved earlier.

The flattening stratification will now be a consequence of the following:

Claim: $\mathcal{F}_B$ is flat over $\mathrm{Spec}(B) \Leftrightarrow g^* f_* \mathcal{F}(d)$ is locally free for all $d \geq d_0$.

If $\mathcal{E}$ is a coherent sheaf on S and the fiber $\mathcal{E}(s)$ at $s \in S$ has rank e, then by Nakayama's lemma there is a presentation of $\mathcal{E}|_U$ in a neighborhood $s \in U$ of the form:

$$(**) \quad \mathcal{O}_U^f \stackrel{\psi_{ij}}{\to} \mathcal{O}_U^e \to \mathcal{E}|_U \to 0$$

The equations $\psi_{ij} = 0$ patch together to put a scheme structure S_e on the locally closed subset $V_e \subset S$ where $\mathcal{E}$ has constant rank e.

It is now easy to see that $g^*\mathcal{E}$ is locally free of rank e if and only if $g(T) \subset V_e$ (as sets) and $g^*\psi_{ij} = 0$ for all ψ_{ij}, which is to say if and only if $g : T \to S$ factors through S_e. Thus we obtain a locally free stratification $S = \coprod S_e$ associated to $\mathcal{E}$ and indexed by the rank of $\mathcal{E}$.

Now assume that the claim is true. Choose $n+1$ to exceed the degree of the Hilbert polynomial of each $\mathcal{F}_s$ in the family. Thus the ranks of the vector spaces $\mathrm{H}^0(X_s, \mathcal{F}_s(d_0)), ..., \mathrm{H}^0(X_s\mathcal{F}_s(d_0+n))$ determine the Hilbert polynomial of each $\mathcal{F}_s$ by (i) above. By (ii) above, these are the fibers of the sheaves $f_*\mathcal{F}(d_0), ..., f_*\mathcal{F}(d_0+n)$ since they are the sheafifications of the A-modules $\mathrm{H}^0(X_A, \mathcal{F}_A(d_0)), ..., \mathrm{H}^0(X_A, \mathcal{F}_A(d_0+n))$.

If we intersect the locally closed subschemes obtained in the locally free stratifications of each of $f_*\mathcal{F}(d_0), f_*\mathcal{F}(d_0+1), ..., f_*\mathcal{F}(d_0+n)$, we obtain a stratification $S = \coprod S_{P(d)}$ indexed by (distinct) Hilbert polynomials with the property that $g^*f_*\mathcal{F}(d_0), ..., g^*f_*\mathcal{F}(d_0+n)$ are simultaneously locally free if and only if $g : T \to S$ factors through $\coprod S_{P(d)}$. Further intersecting with the schemes obtained from the locally free stratifications of the rest of the sheaves $f_*\mathcal{F}(d_0+n+1), ...$ only serves to shrink the scheme structure on $S_{P(d)}$ leaving the same underlying reduced scheme $V_{P(d)}$. Since S is Noetherian, it follows that after finitely many such intersections, we obtain the limit scheme structure which according to the claim is the desired flattening stratification.

By the stable base change result, the maps:

$$g^*f_*\mathcal{F}(d) = \mathrm{H}^0(X, \mathcal{F}) \otimes_A B \to \mathrm{H}^0(X_B, \mathcal{F}_B(d)) = \widetilde{f}_*\widetilde{g}^*\mathcal{F}(d)$$

are isomorphisms for all $d >> 0$. Thus if the $g^*f_*\mathcal{F}(d)$ are all locally free for $d \geq d_0$, then so are the $\widetilde{f}_*\widetilde{g}^*\mathcal{F}(d)$ for all $d >> 0$, and by the equivalence of conditions (b) and (a) in the constancy of the Hilbert polynomial, this imples that $\widetilde{g}^*\mathcal{F}$ is flat over T.

On the other hand, if $t \in \mathrm{Spec}(B)$ maps to $s \in \mathrm{Spec}(A)$, then by (i):

$$0 = \mathrm{H}^i(X_s, \mathcal{F}_s(d)) \otimes_{k(s)} k(t) \xrightarrow{\sim} \mathrm{H}^i(X_t, \mathcal{F}_t(d))$$

for all $d \geq d_0$ and all $i > 0$ because $\mathrm{Spec}(k(t))$ is flat over $\mathrm{Spec}(k(s))$. But these are the higher cohomologies of the fibers of $\widetilde{g}^*\mathcal{F}(d)$. Thus if $\widetilde{g}^*\mathcal{F}$ is flat over T, then by the corollary to cohomology and base change, it follows that $\widetilde{f}_*\widetilde{g}^*\mathcal{F}(d)$ is locally free for all $d \geq d_0$. But now the map $g^*f_*\mathcal{F}(d) \to \widetilde{f}_*\widetilde{g}^*\mathcal{F}(d)$ induces the following isomorphisms fibers:

$$\mathrm{H}^0(X_s, \mathcal{F}_s(d)) \otimes_{k(s)} k(t) \xrightarrow{\sim} \mathrm{H}^0(X_t, \mathcal{F}_t(d))$$

using (ii) and cohomology and base change. Any map from a coherent sheaf to a locally free sheaf which is an isomorphism on fibers is an isomophism, by Nakayama's lemma, so the claim is proved.